LECTURE 8

LAST TIME - LIMITS

\[ \lim_{(x,y) \to (a,b)} f(x,y) = L \]

Given \( \varepsilon > 0 \), there is \( \delta > 0 \), so that

\[ |f(x,y) - L| < \varepsilon \quad \text{if} \quad 0 < |(x-a, y-b)| < \delta. \]

...same for several variables...

LIMIT LAWS

1. \[ \lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \]
2. \[ \lim_{x \to a} f(x)g(x) = \left( \lim_{x \to a} f(x) \right) \left( \lim_{x \to a} g(x) \right) \]
3. \[ \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \]

provided right hand sides exist.

CONTINUITY

\( \mathbb{R} \times (\mathbb{R} \setminus \{a\}) \), \( \mathbb{R} = (a_1, a) \)

\( f(x) \) is continuous at \( \bar{x} \) if

\[ \lim_{x \to \bar{x}} f(x) = f(\bar{x}) \]

(\( f(\bar{x}) \)) approx. \( f(x) \)...

\( f \) is continuous if it is continuous at every point in its domain.

LIMIT LAWS \( \Rightarrow \) can build continuous functions by \( +, -, \times, \div \) continuous.

Can evaluate limits by "plugging in".

DERIVATIVES

Recall derivative at \( a \) for \( f: \mathbb{R} \to \mathbb{R} \):

\[ f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{if limit exists} \]
For functions of several variables, there are a number of notions of derivative we could consider — simplest first.

**Partial Derivatives**

\[ f: \mathbb{R}^2 \to \mathbb{R} \]

- Fix one of the variables to be constant

\[ z = f(x, y) \]

\[ z = f(a, y) \]

Slope in \( y \)-plane = \( \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h} \)

DEF \( \frac{\partial f}{\partial y} (a, b) \)

Similarly, define

\[ \frac{\partial f}{\partial x} (a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} \]
To compute partial derivatives, treat other variables as constant.

\[ \frac{\partial f}{\partial x}(a,b) = \left( \frac{\partial f}{\partial x} \right)_{(a,b)} = f_x(a,b) = D_1 f(a,b) \]

\[ \frac{\partial f}{\partial y}(a,b) = \left( \frac{\partial f}{\partial y} \right)_{(a,b)} = f_y(a,b) = D_2 f(a,b) \]

Higher order partial derivatives:

\[ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = f_{yx} = f_yx \]

\[ \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = f_{xx} \]

Can also make sense of partial derivatives in higher dimensions.

\[ f(x,y,z,w) \]

\[ \frac{\partial f}{\partial z} = \lim_{h \to 0} \frac{f(x,y,z+h,w) - f(x,y,z,w)}{h} \]

Next time - tangent planes