FUNCTIONS FROM $\mathbb{R}^n \to \mathbb{R}^m$

- **Example 1**: 
  \[ f(x) = x^2 - 1 \]
  \[ f: \mathbb{R} \to \mathbb{R} \]
  Domain and range both $\mathbb{R}$.

- **Example 2**: 
  \[ f(x, y) = x^2 + y^2 \]
  \[ f: \mathbb{R}^2 \to \mathbb{R} \]
  Domain is $\mathbb{R}^2$.

- **Example 3**: 
  \[ f(x, y, z) = (xy, yz) \]
  \[ f: \mathbb{R}^3 \to \mathbb{R}^2 \]
  Domain is $\mathbb{R}^3$.

Much of general theory is evident for $f: \mathbb{R}^2 \to \mathbb{R}$, so we start there.

**Graphs**

1. \[ f(x) = x^2 - 1 \]
   - Graph of $f = \{(x, f(x)) \in \mathbb{R}^2 \}$
   
   ![Graph of $f(x) = x^2 - 1$](image)

2. \[ f(x, y) = x^2 + y^2 \]
   - Graph of $f = \{(x, y, f(x, y)) \in \mathbb{R}^3 \}$
   
   ![Graph of $f(x, y) = x^2 + y^2$](image)
Can try to visualize graphs by intersecting with planes:

\[ y = -1, \quad y = 0, \quad y = 1 \]

For varying values of \( c \), gives "slices" of \( \mathbb{R}^3 \).

Intersection with graph \( z = x^2 + y^2 \) is \( z = x^2 + c^2 \), helps visualize entire graph in \( \mathbb{R}^3 \) (reconstruct graph from slices—think MRI).

Intersecting with \( z = c \) is particularly useful.

Then just see \( \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = c^2\} \), so for different values of \( c \):

These are examples of level sets (or level curves in this case, \( f: \mathbb{R}^2 \to \mathbb{R} \))

In general, the level sets of a function \( f: \mathbb{R}^n \to \mathbb{R} \) is \( \{(x_1, \ldots, x_n) \mid f(x_1, \ldots, x_n) = c \} \) for any constant \( c \).
EX \( f(x,y) = y^2 - x^2 \)
\[ y^2 - x^2 = c \text{, for } c = -2, -1, 0, 1, 2 \]
GRAPH IS A SADDLE

EX \( g(x,y) = x + 2y + 1 \)
\[ x + 2y + 1 = c \]
LINES with
SLOPE \(-1\)
EQUALLY SPACED

UP A DIMENSION:
EX \( h(x,y,z) = x^2 + y^2 + z^2 \)
WHAT ARE THE LEVEL SETS?
concentric spheres
\[ x^2 + y^2 + z^2 = c \]
GRAPH IS \( \{ (x,y,z) \mid f(x,y,z) \} \) OR \( x^2 + y^2 + z^2 = 0 \) HARD TO VISUALIZE

EX HOPF FIBRATION (BASICALLY) \( F : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) LEVEL CURVES - SEE VIDEO