Previously

**Theorem** (Green’s Theorem). Let $D$ be a set in $\mathbb{R}^2$, so that $\partial D$ is one or more simple closed paths. Let $\mathbf{F} = \langle P, Q \rangle$ have continuous 1st partial derivatives on $D$. Then

$$
\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_{\partial D} P \, dx + Q \, dy = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}.
$$

Today’s Objectives

1. Compute the divergence of a vector field.
2. Compute the curl of a vector field.
3. Determine if a vector field on $\mathbb{R}^3$ is conservative; consider curl and line integrals.
4. Determine if a vector field on $\mathbb{R}^3$ is the curl of some other field; consider divergence.
Green’s Theorem

Draw blob with small hole. $C_1$ wraps counterclockwise around the hole, and $C_2$ and $C_3$ are paths from $P$ to $Q$ so that $C_3$ plus $-C_2$ wraps counterclockwise around hole.

Question 1. Let $\mathbf{F} = \langle P, Q \rangle$ be a vector field on the set $D$ above with continuous 1st partial derivatives. Assume that $P_y = Q_x$, $\int_{C_1} \mathbf{F} \cdot \mathbf{dr} = 3$, and $\int_{C_2} \mathbf{F} \cdot \mathbf{dr} = -1$. Use Green’s theorem to find $\int_{C_3} \mathbf{F} \cdot \mathbf{dr}$.

(A) 0
(B) 2
(C) 3
(D) 4
(E) We don’t have enough information.

Hint: Find a set $B$ in $D$ so that $\partial B$ lies on the $C_i$.

There is a set $B$ in $D$ so that $\partial B = -C_1, -C_2, \text{ and } C_3$. $\mathbf{F}$ is defined on $B$, so

$0 = \iint_B \nabla \cdot \mathbf{F} \, dA = \iint_B (Q_x - P_y) \, dA$

$= \iiint_B \mathbf{F} \cdot \mathbf{dr} = -\int_{C_1} \mathbf{F} \cdot \mathbf{dr} - \int_{C_2} \mathbf{F} \cdot \mathbf{dr} + \int_{C_3} \mathbf{F} \cdot \mathbf{dr}$

$= -3 - (-1) \cdot \int_{C_3} \mathbf{F} \cdot \mathbf{dr}$
Curl

Notation:
\[ \partial_x = \frac{\partial}{\partial x}, \quad \partial_y = \frac{\partial}{\partial y}, \quad \partial_z = \frac{\partial}{\partial z} \quad \& \quad \nabla = (\partial_x, \partial_y, \partial_z) \]

Given a vector field \( \mathbf{F} = (P, Q, R) \) on a set in \( \mathbb{R}^3 \), define the curl of \( \mathbf{F} \) by

\[
\text{curl } \mathbf{F} := \nabla \times \mathbf{F} = \begin{vmatrix}
    \mathbf{i} & \mathbf{j} & \mathbf{k} \\
    \partial_x & \partial_y & \partial_z \\
    P & Q & R
\end{vmatrix} = (R_y - Q_z, P_z - R_x, Q_x - P_y).
\]

Example. Find \( \text{curl} \langle e^x, \sin y, xyz \rangle \)

\[
\text{curl } \langle e^x, \sin y, xyz \rangle = \begin{vmatrix}
    \hat{e}_x & \hat{e}_y & \hat{e}_z \\
    \partial_x & \partial_y & \partial_z \\
    e^x & \sin y & xyz
\end{vmatrix} = \langle xz, -yz, 0 \rangle.
\]

Example. Find \( \text{curl} \langle P(x, y), Q(x, y), 0 \rangle \).

\[
\text{curl } \langle P(x, y), Q(x, y), 0 \rangle = \langle 0, 0, Q_x - P_y \rangle.
\]

Where have we seen \( Q_x - P_y \)?

If the vector field \( \mathbf{F} = (P, Q) \) is conservative, \( Q_x - P_y = 0 \), i.e. the two-dimensional curl vanishes.

The converse holds if the set is simply connected.

Also, the integral of \( \mathbf{F} \) along the boundary of \( D \) is the integral of \( Q_x - P_y \) over \( D \) itself.
Given a vector field $\mathbf{F} = \langle P, Q, R \rangle$ on a set in $\mathbb{R}^3$, define
\[
\text{curl } \mathbf{F} := \nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle
\]

**Question 2.** Compute $\mathbf{G} = \text{curl} \langle e^x, z \cos y, \sin y \rangle$.

*How many components of $\mathbf{G}$ are 0?*

(A) All three, i.e. $\mathbf{G} = \mathbf{0}$.

(B) Two.

(C) One.

(D) None.

(E) I don’t know.

\[
\text{curl } \langle e^x, z \cos y, \sin y \rangle = \begin{vmatrix}
1 & 1 & k \\
e^x & e^y & e^z \\
e^x & z \cos y & \sin y
\end{vmatrix}
\]

\[
= \langle e^x \cos y - e^x \cos y, 0, 0 \rangle = \langle 0, 0, 0 \rangle
\]
Theorem. Let $f$ be a function on an open set $D$ in $\mathbb{R}^3$ with continuous 2nd partial derivatives. Then

$$\text{curl}(\nabla f) = 0.$$ 

Proof.

Let $\mathbf{F}$ be a vector field on an open set $D$ in $\mathbb{R}^3$ with continuous 1st partial derivatives. By the theorem above, if $\mathbf{F}$ is conservative, then $\text{curl} \mathbf{F} = 0$. The converse is true if $D = \mathbb{R}^3$.

Theorem. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ with continuous 1st partial derivatives. If $\text{curl} \mathbf{F} = 0$, then $\mathbf{F}$ is conservative.

The proof is similar to the 2-dimensional case; we'll do it the last day.

The flow charts for deciding if a vector field is conservative are analogous in dimension 2 and 3, except:

- in dimension 2 we check if $Q_y - P_x = 0$, but in 3 we check if $\text{curl} \mathbf{F} = 0$ &
- in dimension 2 we check if the domain is simply connected, but in 3 we check if it's $\mathbb{R}^3$.

Question 3. Is $\mathbf{F} = (e^x, \sin y, xyz)$ conservative?

Is $\mathbf{G} = (e^x, z \cos y, \sin y)$?

(A) Yes & Yes
(B) Yes & No
(C) No & Yes
(D) No & No
(E) I don’t know (yet).

$\mathbf{F}$ is not conservative, because $\text{curl} \mathbf{F} \neq 0$.

$\mathbf{G}$ is conservative, because $\text{curl} \mathbf{G} = 0$ and it’s defined on $\mathbb{R}^3$. 

**Divergence**

Given a vector field $\mathbf{F} = \langle P, Q, R \rangle$ on a domain in $\mathbb{R}^3$, define the **divergence** of $\mathbf{F}$ by

$$\text{div} \, \mathbf{F} := \nabla \cdot \mathbf{F} = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z$$

**Theorem.** Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field on an open set $D$ in $\mathbb{R}^3$ with continuous 2nd partial derivatives. Then

$$\text{div} (\text{curl} \, \mathbf{F}) = 0$$

**Proof.**

Clairaut’s thm+ calculation.

**Example.** Does there exist $\mathbf{G}$ with $\text{curl} \, \mathbf{G} = \langle x, y, z \rangle$?

$$\text{div} \, \langle x, y, z \rangle = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 1 + 1 + 1 = 3$$

For $\langle x, y, z \rangle = \text{curl} \, \mathbf{G}$, then

$$0 = \text{div} (\text{curl} \, \mathbf{G}) = \text{div} \, \langle x, y, z \rangle = 3 \rightarrow \text{false}$$

So $\langle x, y, z \rangle$ is NOT $\text{curl} \, \mathbf{G}$ for any $\mathbf{G}$.
The big picture

Optional

\{\text{constants}\} \rightarrow \{\text{functions}\} \xrightarrow{\nabla} \{\text{vector fields}\} \\
\xrightarrow{\text{curl}} \{\text{vector fields}\} \xrightarrow{\text{div}} \{\text{functions}\}

Theorems:
(1) $\nabla(\text{constant}) = 0$
(2) $\text{curl}(\nabla(f)) = 0$
(3) $\text{div}(\text{curl}(F)) = 0$

Any two arrows in a row give zero.

Other combinations either
\begin{itemize}
  \item don’t make sense, or
  \item don’t give zero.
\end{itemize}

Example. The Laplacian is $\nabla^2 f := \text{div}(\nabla f)$. Find $\nabla^2 x^2$.

Cool math:
If the domain $D$ is $\mathbb{R}^3$, the converse is true:
(1) If $\nabla f = 0$, then $f$ is a constant.
(2) If $\text{curl} \ F = 0$, then $\ F = \nabla f$ for some $f$.
(3) If $\text{div} \ F = 0$, then $\ F = \text{curl} \ G$ for some $G$.

For general $D \subset \mathbb{R}^3$, these claims fail.
We can use this failure to study subsets of $\mathbb{R}^3$. 