GREEN'S THEOREM

\[ \mathbf{F} = (e^{-x+y}, e^{y}) \]

\[ \oint_{C} \mathbf{F} \cdot d\mathbf{T} = \int_{0}^{\pi/2} \left( e^{-t+0}, e^{0} \right) \cdot (1,0) \, dt - \int_{0}^{\pi/2} \left( e^{t+\cos t}, e^{\sin t} \right) \cdot (\sin t, \cos t) \, dt \]

\[ \left( x(t), y(t) \right) = (t, 0) \quad t \in [0, \pi/2] \]

\[ \mathbf{F} \cdot \mathbf{T} = \int_{0}^{\pi/2} \left( e^{-t} - e^{t+\cos t - \sin t \cos t} \right) \, dt \]

\[ = \int_{0}^{\pi/2} e^{-t} \, dt - \int_{0}^{\pi/2} e^{t+\cos t - \sin t \cos t} \, dt \]

\[ = \int_{0}^{\pi/2} \sin t e^{\cos t} \, dt - \int_{0}^{\pi/2} \cos t = \left. e^{u} du - \sin t \right|_{0}^{\pi/2} = -(1+1) = -2 \]

\[ u = \cos t \]
\[ du = -\sin t \, dt \]

GREEN'S THEOREM IS NEXT FORM OF FUNDAMENTAL THEOREM.

C A SIMPLE CLOSED CURVE IN IR²

POSITIVE ORIENTATION IS COUNTERCLOCKWISE ORIENTATION.

[Diagram of positive and negative orientations]
Write \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) for integral over \( C \) with positive orientation.

\( C \) is the boundary of a set \( D \) in \( \mathbb{R}^2 \), we also write \( \partial D = C \) with positive orientation. Thus:

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\partial D} \mathbf{F} \cdot d\mathbf{S}.
\]

**Green's Theorem**: \( C \) simple closed curve (piecewise \( C^1 \)) bounding \( D \), \( \mathbf{F} = (P,Q) \) w/ \( P, Q \) \( C^1 \) functions. Then:

\[
\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}.
\]

Think: "derivative" of \( \mathbf{F} \).

*Ex: Calculate problem again:*

\[
\oint_C (e^{-x+y}, e^y) \cdot d\mathbf{r} = \iint_D 0 - (1) \, dA = \iint_D -1 \, dA = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dy \, dx = \int_{-\pi}^{\pi} \sin(\theta) \, d\theta = -(1+1) = -2.
\]

*Ex: \( \mathbf{F} \) conservative (or \( Q_x - P_y \))

Then:

\[
\iint_D \mathbf{F} \cdot d\mathbf{S} = \iint_D 0 \, dA = 0,
\]

path independence is a special case of Green's Theorem.
Q. WHY IS GREEN'S THM TRUE?
A. FUND. THM OF CALC!

SPECIAL CASE

\[ C = \begin{array}{ccc}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
\end{array} \]

\[ \int (P,Q) \cdot dx = \int_{C_1} (P,Q) \cdot dx + \int_{C_2} (P,Q) \cdot dx + \int_{C_3} (P,Q) \cdot dx + \int_{C_4} (P,Q) \cdot dx \]

\[ = \int_a^b P(x,c) \, dx - \int_a^b P_x(x,d) \, dx - \int_c^d Q(x,y) \, dy + \int_c^d Q_y(x,y) \, dy \]

\[ = \mathbf{F} \mathbf{C} \int_a^b \int_c^d -P_y(x,y) \, dy \, dx + \int_c^d Q_x(x,y) \, dx \, dy \]

\[ = \int_a^b \int_c^d Q_x - P_y \, dA \]

\[ \text{EX CONSIDER } \mathbf{F} = \frac{1}{2}(-y,x) \]

\[ \int \mathbf{F} \cdot d\mathbf{r} = \iint_D \frac{1}{2}(1-(-1)) \, dA = \iint_D dA = \text{AREA OF } D \]

ALSO WORKS FOR U.F. (-y,0) ON (a,x).

==SPECIAL FOR THESE VECTOR FIELDS!==
GENERAL FORM OF GREEN'S THEOREM

D A CLOSED BOUNDED SET BOUNDED BY MULTIPLE SIMPLE CLOSED CURVES.

$\partial D =$ UNION OF THE CURVES ORIENTED SO $D$ IS "ON THE LEFT" OF THE CURVES

GREEN'S THEOREM $\mathbf{F} = (P, Q)$ C¹ VECTOR FIELD

$$\iint_D Q_x - P_y \, dA = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{k} \oint_{C_i} \mathbf{F} \cdot d\mathbf{s}$$

WHERE $\partial D = C_1 \cup \ldots \cup C_k$ AS ORIENTED CURVE.

EX $\mathbf{F}(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ C ANY SIMPLE CLOSED CURVE SURROUNDING (0,0)

CLAIM: $\oint_C \mathbf{F} \cdot d\mathbf{s} = 2\pi$ 

TAKE A TINY CIRCLE CENTERED AT (0,0) \ radius $r > 0$ \ D REGION BETWEEN C & $C_r$ THEN

$0 = \iint_D Q_x - P_y \, dA = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \oint_C \mathbf{F} \cdot d\mathbf{s} - \oint_{C_r} \mathbf{F} \cdot d\mathbf{s} = \oint_{C_r} \mathbf{F} \cdot d\mathbf{s} - 2\pi$. 