Lecture 22  Double Integrals §15.1, 15.2

So far, all integrals have been "1-dimensional!"
- Integrals over intervals or curves... not any more!

Problem

\[ \mathcal{E} = x^2 + y + 1 \]

Volume?

\[ R = \{ (x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1 \} \]

Approximate by volumes of boxes:

\[ R_{ij} = \{ (x, y) | x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j \} \]

\[ \Delta x \]

\[ \Delta y \]

Area \Delta x \Delta y

Have subdivision into rectangles

\[ R_{ij} = \{ (x, y) | x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j \} \]
Pick a point \((x_i^*, y_j^*)\) in \(R_{ij}\).

\[
\text{Volume} \approx \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*) \Delta x \Delta y
\]

(volume of box above \(R_{ij}\))

with height given by value of \(f\)
at some point.

\[
\int f \text{ continuous} \Rightarrow \text{any two heights above } R_{ij} \text{ are about the same.}
\]

Take a limit as \(|\Delta x|, |\Delta y| \to 0\)

\[
\lim_{\Delta x, \Delta y \to 0} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*) \Delta x \Delta y = \iint_{R} f(x,y) \, dA
\]

(area element).

Q: HOW DO WE CALCULATE THIS?

A: SLICE!

\[
\text{Volume} = \iint_{R} f(x,y) \, dA = \int_{a}^{b} A(x) \, dx = \int_{a}^{c} \left( \int_{a}^{b} f(x,y) \, dy \right) \, dx
\]

Similarly,

\[
\iint_{R} f(x,y) \, dA = \iint_{c}^{d} f(x,y) \, dx \, dy
\]

\(x\) constant for inner integral.

so, can now integral function of one variable

\(\text{DOUBLE INTEGRAL} = \text{ITERATED INTEGRAL}\)

FUBINI THEOREM
Exercise: Volume above unit square \( R = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\} \). Below graph \( z = f(x,y) = x^2y + y^2 + x + 4 \).

\[
\iiint_R x^2y + y^2 + x + 4 \, dA = \iint_0^1 x^2y + y^2 + x + 4 \, dx \, dy
\]

\[
= \int_0^1 \left( \frac{x^3y}{3} + xy^2 + \frac{x^2}{2} + 4x \right) \, dy
\]

\[
= \int_0^1 \frac{y}{3} + y^2 + \frac{x}{2} + 4 \, dy = \frac{y^2}{6} + \frac{y^3}{3} + \frac{9y}{2} \bigg|_0^1
\]

\[
= \frac{1}{6} + \frac{1}{3} + \frac{9}{2} = \frac{30}{6} = 5 \quad \square
\]

Original Example:

\( R = [0,1] \times [0,1] \), \( f(x,y) = x^2 + y^2 + 1 \).

\[
\iiint_R x^2 + y^2 + 1 \, dA = \iint_0^1 x^2 + y^2 + 1 \, dx \, dy = \int_0^1 \frac{x^3}{3} + xy^2 + x \bigg|_0^1 \, dy
\]

\[
= \int_0^1 \frac{1}{3} + y^2 + 1 \, dy = \frac{y^3}{3} + y^3 \bigg|_0^1 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.
\]

Sometimes it's easier to calculate in one order versus the other.

\( R = [0,2] \times [0,1] \).

\[
\iiint_R xe^{xy} \, dA = \iint_0^2 xe^{xy} \, dy \, dx = \int_0^2 x(e^{x} - 1) \, dx = \int_0^2 e^{3x} - e^{-x} \, dx
\]

\[
= e^{\frac{3x}{3}} - e^{-x} \bigg|_0^2 = e^6 - e^{-2} - \frac{1}{3}
\]

Try this in the other order—it's less fun!