1. Consider the vector field \( \mathbf{F} = (y, 0) \) on \( \mathbb{R}^2 \).

(a) Draw a sketch of \( \mathbf{F} \) on the region where \(-2 \leq x \leq 2 \) and \(-2 \leq y \leq 2 \). Check your answer with the instructor.

**SOLUTION:**

Below is the image for parts (a) and (b)

(b) Consider the following two curves which start at \( A = (-2, 0) \) and end at \( B = (2, 0) \), namely the line segment \( C_1 \) and upper semicircle \( C_2 \).

Add these curves to your sketch, and compute both \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} \) and \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \). Check your answers with the instructor.

**SOLUTION:**

Parametrize \( C_1 \) by \( \mathbf{r}_1(t) = (t, 0), 0 \leq t \leq 2 \) and parametrize \( C_2 \) by \( \mathbf{r}_2(t) = (-2 \cos t, 2 \sin t), 0 \leq t \leq \pi \).

We have

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 F(\mathbf{r}_1(t)) \cdot \mathbf{r}'_1(t) \, dt = \int_0^2 (0, 0) \cdot (1, 0) \, dt = 0
\]

\[
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi F(\mathbf{r}_2(t)) \cdot \mathbf{r}'_2(t) \, dt = \int_0^\pi (2 \sin t, 0) \cdot (2 \sin t, 2 \cos t) \, dt = 4 \int_0^\pi \sin^2 t \, dt
\]

\[
= 4 \cdot \frac{1}{2} \left[ t - \frac{1}{2} \sin(2t) \right]_0^\pi = 2\pi
\]

(c) Based on your answer in (b), could \( \mathbf{F} \) be \( \nabla f \) for some \( f : \mathbb{R}^2 \to \mathbb{R} \)? Explain why or why not.

**SOLUTION:**

By the Fundamental Theorem of Line Integrals, if \( \mathbf{F} = \nabla f \) for some \( f : \mathbb{R}^2 \to \mathbb{R} \) then \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is path independent for any curve \( C \) starting at \( A = (-2, 0) \) and ending at \( B = (2, 0) \). Since we obtained different answers for the paths \( C_1 \) and \( C_2 \), \( \mathbf{F} \) cannot be of this form.
2. Consider the curve C and vector field \( \mathbf{F} \) shown below.

(a) Calculate \( \mathbf{F} \cdot \mathbf{T} \), where here \( \mathbf{T} \) is the unit tangent vector along \( C \). Without parameterizing \( C \), evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) by using the fact that it is equal to \( \int_C \mathbf{F} \cdot \mathbf{T} \, ds \).

**SOLUTION:**

From the picture we suppose that \( \mathbf{F}(x, y) = (1, 1) \). We have \( \mathbf{T} = \frac{1}{\sqrt{5}}(-2, -1) \), so \( \mathbf{F} \cdot \mathbf{T} = \frac{-3}{\sqrt{5}} \). So

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \frac{-3}{\sqrt{5}} \int_C ds = -3
\]

since \( \int_C ds \) is simply the distance between \((1,1)\) and \((3,2)\).

(b) Find a parameterization of \( C \) and a formula for \( \mathbf{F} \). Use them to check your answer in (a) by computing \( \int_C \mathbf{F} \cdot d\mathbf{r} \) explicitly.

**SOLUTION:**

Parametrize \( C \) by \( \mathbf{r}(t) = (3 - 2t, 2 - t), 0 \leq t \leq 1 \). We already have \( \mathbf{F} = (1, 1) \). So

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (1, 1) \cdot (-2, -1) \, dt = -3
\]

3. Consider the points \( A = (0,0) \) and \( B = (\pi, -2) \). Suppose an object of mass \( m \) moves from \( A \) to \( B \) and experiences the constant force \( \mathbf{F} = -mg \mathbf{j} \), where \( g \) is the gravitational constant.

(a) If the object follows the straight line from \( A \) to \( B \), calculate the work \( W \) done by gravity using the formula from the first week of class.

**SOLUTION:**

Recall that the work done on an object moving along a straight line subject to a constant force \( \mathbf{F} \) is \( W = \mathbf{F} \cdot \mathbf{D} \), where \( \mathbf{D} \) is the displacement vector. In this case \( \mathbf{D} = (\pi, -2) \) and \( \mathbf{F} = (0, -mg) \). So

\[
W = (\pi, -2) \cdot (0, -mg) = 2mg.
\]
(b) Now suppose the object follows half of an inverted cycloid $C$ as shown below. Explicitly parameterize $C$ and use that to calculate the work done via a line integral.

![Graph of an inverted cycloid with points](image)

**SOLUTION:**

A parametrization for the inverted cycloid $C$ is $\mathbf{r}(t) = (t - \sin t, \cos t - 1), 0 \leq t \leq \pi$. So

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (0, -mg) \cdot (1 - \cos t, -\sin t) \, dt = \int_0^\pi mg \sin t \, dt = mg \left[ -\cos t \right]_0^\pi = 2mg$$

(c) Find a function $f: \mathbb{R}^2 \to \mathbb{R}$ so that $\nabla f = \mathbf{F}$. Use the Fundamental Theorem of Line Integrals to check your answers for (a) and (b). Have you seen the quantity $-f$ anywhere before? If so, what was its name?

**SOLUTION:**

If such an $f$ exists, we must have $f_x = 0$ and $f_y = -mg$. Integrating $-mg$ with respect to $y$ we obtain $f = -mg y + C(x)$, where $C(x)$ is some function of $x$. Differentiating this with respect to $x$ we obtain $f_x = C'(x) = 0$, so $f = -mg y + K$, where $K$ is a constant, is a potential function for $\mathbf{F}$.

By the Fundamental Theorem of Line Integrals, both (a) and (b) must have the same answer, namely

$$\int_L \mathbf{F} \cdot d\mathbf{r} = \int_L \nabla(f) \cdot d\mathbf{r} = f(B) - f(A) = f(\pi, -2) - f(0, 0) = (-mg(-2) + K) - K = 2mg$$

where $L$ is the line segment from $A$ to $B$ and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla(f) \cdot d\mathbf{r} = f(B) - f(A) = 2mg$$

The quantity $-f$ is called the potential energy.
4. If you get this far, work #52 from Section 16.2:

48. Experiments show that a steady current $I$ in a long wire produces a magnetic field $\mathbf{B}$ that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as in the figure). Ampère’s Law relates the electric current to its magnetic effects and states that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where $I$ is the net current that passes through any surface bounded by a closed curve $C$, and $\mu_0$ is a constant called the permeability of free space. By taking $C$ to be a circle with radius $r$, show that the magnitude $B = |\mathbf{B}|$ of the magnetic field at a distance $r$ from the center of the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$

SOLUTION:

We are assuming that $\mathbf{B}$ has magnitude which only depends on the distance from the wire. So $B = |\mathbf{B}|$ is constant along any circle centered around the wire in a plane perpendicular to the wire. Let $C = r(t)$ be such a circle with radius $r$ parametrized in the counterclockwise direction and let $B$ denote the magnitude of $\mathbf{B}$ along $C$. Note that $\mathbf{B}(r(t))$ is a positive multiple of $r'(t)$ by definition. So it follows that $\mathbf{T}(t)$, the unit tangent vector to $C$, is given by $\mathbf{T}(t) = \frac{\mathbf{B}(r(t))}{B}$. We have

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \int_C \mathbf{B} \cdot \mathbf{T} ds = \int_C \frac{\mathbf{B} \cdot \mathbf{B}}{B} ds = B \int_C ds = 2\pi r B$$

By Ampere's Law, $\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$, so we have $2\pi r B = \mu_0 I$, or $B = \frac{\mu_0 I}{2\pi r}$. 