1. Consider the curve \( C \) in \( \mathbb{R}^3 \) given by
\[
\mathbf{r}(t) = \left( e^t \cos t \right) \mathbf{i} + 2 \mathbf{j} + \left( e^t \sin t \right) \mathbf{k}
\]

(a) Draw a sketch of \( C \).

**Solution.** The sketch of \( C \) is the following graph.

![Figure 1: Sketch of C.](image)

(b) Calculate the arc length function \( s(t) \), which gives the length of the segment of \( C \) between \( \mathbf{r}(0) \) and \( \mathbf{r}(t) \) as a function of the time \( t \) for all \( t \geq 0 \). Check your answer with the instructor.

**Solution.** Since
\[
\begin{align*}
x'(t) &= e^t \cos t - e^t \sin t, \\
y'(t) &= 0, \\
z'(t) &= e^t \sin t + e^t \cos t,
\end{align*}
\]
we have
\[
|\mathbf{r}'(t)| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} = \sqrt{2} e^t.
\]
Hence the arc length is
\[
s(t) = \int_0^t |\mathbf{r}'(u)| \, du = \int_0^t \sqrt{2} e^u \, du = \sqrt{2} e^t - \sqrt{2}.
\]
(c) Now invert this function to find the inverse function \( t(s) \). This gives time as a function of arclength, that is, tells how long you must travel to go a certain distance.

**Solution.** Solve \( s = \sqrt{2} e^t - \sqrt{2} \), which gives \( e^t = \frac{s + \sqrt{2}}{\sqrt{2}} \), and so

\[
t = t(s) = \ln \left( \frac{s + \sqrt{2}}{\sqrt{2}} \right).
\]

(d) Suppose \( h : \mathbb{R} \to \mathbb{R} \) is a function. We can get another parameterization of \( C \) by considering the composition

\[
f(s) = r(h(s))
\]

This is called a *reparametrization*. Find a choice of \( h \) so that

i. \( f(0) = r(0) \)

ii. The length of the segment of \( C \) between \( f(0) \) and \( f(s) \) is \( s \). (This is called parametrizing by arc length.)

Check your answer with the instructor.

**Solution.** From (c) we know \( t = \ln \left( \frac{s + \sqrt{2}}{\sqrt{2}} \right) \). When \( s = 0 \), we have \( t = \ln 1 = 0 \). Then we can choose

\[
h(s) = \ln \left( \frac{s + \sqrt{2}}{\sqrt{2}} \right).
\]

(e) Without calculating anything, what is \( |f'(s)| \)?

**Solution.** Since \( s = \int_0^s |f'(u)| \, du \), then by the fundamental theorem of calculus, we can differentiate both sides with respect to \( s \) and get \( 1 = |f'(s)| \).

2. Consider the curve \( C \) given by the parametrization \( r : \mathbb{R} \to \mathbb{R}^3 \) where \( r(t) = (\sin t, \cos t, \sin^2 t) \).

(a) Show that \( C \) is in the intersection of the surfaces \( z = x^2 \) and \( x^2 + y^2 = 1 \).

**Solution.** Since \( x = \sin t, y = \cos t, z = \sin^2 t \), it is very easy to check that \( z = x^2 \) and \( x^2 + y^2 = 1 \). So the curve \( C \) lies in both these two surfaces, hence is in the intersection of them.

(b) Use (a) to help you sketch the curve \( C \).

**Solution.** The left graph is the intersection of the two surfaces, while the right one is the curve.
3. (a) Sketch the top half of the sphere $x^2 + y^2 + z^2 = 5$. Check that $P = (1, 1, \sqrt{3})$ is on this sphere and add this point to your picture.

**Solution.** The top half of the sphere is shown in Figure 3 (the black dot is $P$). Since $1^2 + 1^2 + (\sqrt{3})^2 = 5$, we know $P$ is on this sphere.

(b) Find a function $f(x, y)$ whose graph is the top-half of the sphere. Hint: solve for $z$.

**Solution.** Since $x^2 + y^2 + z^2 = 5$, we have $z^2 = 5 - x^2 - y^2$, and so $z = \pm \sqrt{5 - x^2 - y^2}$. As we only want the top half of the sphere, we can let $f(x, y) = \sqrt{5 - x^2 - y^2}$.

(c) Imagine an ant walking along the surface of the sphere. It walks down the sphere along
the path $C$ that passes through the point $P$ in the direction parallel to the $yz$-plane. Draw this path in your picture.

**Solution.** The black curve in Figure 3 is the path.

(d) Find a parametrization $\mathbf{r}(t)$ of the ant’s path along the portion of the sphere shown in your picture. Specify the domain for $\mathbf{r}$, i.e. the initial time when the ant is at $P$ and the final time when it hits the $xy$-plane.

**Solution.** $x = 1$ along the path and $f(1, y) = \sqrt{4 - y^2}$, so setting $y = t$ we get the parametrization

\[ \mathbf{r}(t) = (1, t, \sqrt{4 - t^2}). \]

4. As in 1(d), consider a reparametrization

\[ \mathbf{f}(s) = \mathbf{r}(h(s)) \]

of an arbitrary vector-valued function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$. Use the chain rule to calculate $|\mathbf{f}'(s)|$ in terms of $\mathbf{r}'$ and $h'$.

**Solution.** By the chain rule, $\mathbf{f}'(s) = \mathbf{r}'(h(s)) \cdot h'(s)$. Taking magnitudes of both sides we have

\[ |\mathbf{f}'(s)| = |\mathbf{r}'(h(s))| \cdot |h'(s)|. \]