Thursday, September 25  *  Solutions  *  Taylor series, the 2nd derivative test, and changing coordinates.

1. Consider \( f(x, y) = 2 \cos x - y^2 + e^{xy} \).

(a) Show that \((0,0)\) is a critical point for \( f \).

SOLUTION:
\[
\frac{\partial f}{\partial x}|_{(0,0)} = (-2 \sin x + ye^{xy})|_{(0,0)} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}|_{(0,0)} = (-2y + xe^{xy})|_{(0,0)} = 0
\]

(b) Calculate each of \( f_{xx}, f_{xy}, f_{yy} \) at \((0,0)\) and use this to write out the 2nd-order Taylor approximation for \( f \) at \((0,0)\).

SOLUTION:
The second order Taylor approximation of a function \( f(x, y) \) at \((0,0)\) is given by
\[
T_2(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{f_{xx}(0, 0)}{2}x^2 + \frac{f_{xy}(0, 0) + f_{yx}(0, 0)}{2}xy + \frac{f_{yy}(0, 0)}{2}y^2.
\]
For this problem we have \( f_{xx} = -2 \cos x + ye^{xy}, f_{yy} = -2 + x^2 e^{xy}, \) and \( f_{xy} = e^{xy} + xy e^{xy} \). So \( f_{xx}(0, 0) = -2 = f_{yy}(0, 0) \) and \( f_{xy}(0, 0) = 1 \). Also \( f(0, 0) = 3 \). So the second order Taylor approximation for \( f \) at \((0,0)\) is \( g(x, y) = 3 - x^2 - y^2 + xy \).

2. Let \( g(x, y) \) be the approximation you obtained for \( f(x, y) \) near \((0,0)\) in 1(b). It’s not clear from the formula whether \( g \), and hence \( f \), has a min, max, or a saddle at \((0,0)\). Test along several lines until you are convinced you’ve determined which type it is. In the next problem, you’ll confirm your answer in two ways.

SOLUTION:
Let’s test a general line \( y = mx \) which goes through \((0,0)\) as \( x \to 0 \). Then \( g(x, mx) = 3 - x^2 - m^2 x^2 + mx^2 = 3 - (1 - m + m^2)x^2 \). The polynomial \( 1 - m + m^2 \) is always positive (it opens upward and has its global minimum at \( m = 1/2 \) where \( 1 - m + m^2 > 0 \)). So \( g(x, mx) \) is always a downward opening parabola. This suggests that \((0,0)\) is a relative maximum.

3. Consider alternate coordinates \((u, v)\) on \( \mathbb{R}^2 \) given by \( (x, y) = (u - v, u + v) \).

(a) Sketch the \( u \)- and \( v \)-axes relative to the usual \( x \)- and \( y \)-axes, and draw the points whose \((u, v)\)-coordinates are: \((-1,2), (1,1), (1,-1)\).

SOLUTION:
If we express \( u \) and \( v \) in terms of \( x \) and \( y \) we get \( u = 1/2(x + y) \) and \( v = 1/2(y - x) \). So the \( u \)-axis is given in \( x \) and \( y \) coordinates by all multiples of the vector \((1,1)\) and the \( v \)-axis is given by all multiples of the vector \((-1,1)\). The two axes and the points are shown below.
(b) Express $g$ as a function of $u$ and $v$, and expand and simplify the resulting expression.

**SOLUTION:**

$$3 - x^2 - y^2 + xy = 3 - (u - v)^2 - (u + v)^2 + (u - v)(u + v) = 3 - (u^2 - 2uv + v^2) - (u^2 + 2uv + v^2) + u^2 - v^2 = 3 - u^2 - 3v^2.$$  

(c) Explain why your answer in 3(b) confirms your answer in 2.

**SOLUTION:**

This is an elliptic paraboloid (in $uv$ coordinates) opening downward with maximum at $(0, 0, 3)$, so it confirms that $(0,0)$ is a local maximum ( $(0,0)$ goes to $(0,0)$ under the transformation, so this reasoning makes sense).

(d) Sketch a few level sets for $g$. What do the level sets of $f$ look like near $(0, 0)$?

**SOLUTION:** The level sets are sketched for $g = 2.7, 2.8, 2.9$ on the left and for $f = 2.7, 2.8, 2.9$ on the right. The level sets for $g$ are ellipses that approximate the level sets of $f$ close to $(0,0)$. The ellipses shrink as they get closer to $g(x,y) = 3$, which consists of the single solution $(x,y) = (0,0)$.

(e) It turns out that there is always a similar change of coordinates so that the Taylor series of a function $f$ which has a critical point at $(0,0)$ looks like $f(u,v) \approx f(0,0) + au^2 + bv^2$. In fact this is why the 2nd derivative test works.

Double check your answer in 2 by applying the 2nd-derivative test directly to $f$.

**SOLUTION:**
The Hessian \( f_{xx}f_{yy} - (f_{xy})^2 \) is \((-2)(-2) - 1^2 = 3 > 0\) at \((0, 0)\) and \(f_{xx}(0, 0) = -2 < 0\). So \( f \) has a relative maximum at \((0, 0)\) as suspected.

4. Consider the function \( f(x, y) = 3xe^y - x^3 - e^{3y} \).

   (a) Check that \( f \) has only one critical point, which is a local maximum.

   **SOLUTION:**
   \[ f_x = 3e^y - 3x^2 \quad \text{and} \quad f_y = 3xe^y - 3e^{3y} \]
   \( f_y = 0 \) only if \( x = e^{2y} \) and \( f_x = 0 \) only if \( e^y = x^2 \).
   Solving these simultaneously we see that \( x \) must satisfy \((x^2)^2 = (e^y)^2 = x\), so \( x = 0, -1, \) or \( 1 \). But \( x = e^{2y} > 0 \) so the only critical point is \( x = 1, y = 0 \). Calculating, we see that \( f_{xx}(1, 0) = f_{yy}(1, 0) = -6 \) and \( f_{xy}(1, 0) = 3 \). So the Hessian \( f_{xx}f_{yy} - (f_{xy})^2 = 36 - 9 = 27 > 0 \) at \((1, 0)\). Since \( f_{xx}(1, 0) < 0 \), the second derivative test tells us that \( f(1, 0) = 1 \) is a local maximum.

   (b) Does \( f \) have an absolute maxima? Why or why not?

   **SOLUTION:**
   \( f \) does not have an absolute maximum. For instance if we take the trace curve \( y = 0 \) we get \( f(x, 0) = 3x - x^3 - 1 \), which is unbounded as \( x \to \infty \). Absolute maxima and minima are only guaranteed over a closed and bounded set in the domain. The plane \( \mathbb{R}^2 \) is closed but not bounded, so there is no guarantee that a continuous function will achieve an absolute maximum or minimum over \( \mathbb{R}^2 \).