Extra notes lecture 6

\[ \lim_{(x,y) \to (0,0)} \frac{\sin^2(xy)}{x^4 + y^2} \text{ Does not exist.} \]

Approach along line \( y = x \):
\[
\lim_{x \to 0} \frac{\sin^2(x^2)}{x^4 + x^2} = \lim_{x \to 0} \frac{\sin^2(x)}{x^2 + 1} = \lim_{x \to 0} \frac{\sin^2(x)}{x^2} \cdot \lim_{x \to 0} \frac{x}{x^2 + 1} = 1 \cdot 0 = 0
\]

... along parabola \( y = x^2 \):
\[
\lim_{x \to 0} \frac{\sin^2(x^2)}{x^4 + x^2} = \lim_{x \to 0} \frac{\sin^2(x)}{x^2} = \frac{1}{2} \neq 0
\]

Two different paths give two different limits, so limit doesn't exist.

Example
\[
\lim_{(x,y,z) \to (0,1,1)} \frac{x + y^2}{w + 1} = \frac{\lim x + \lim y}{\lim w + 1} = \frac{0 + 1}{1 + 1} = \frac{1}{2}.
\]

Continuous function:

Is \( f \) continuous/discontinuous why?

Not at \((a,b)\) b/c limit doesn't exist
Not at \((p,q)\) b/c limit \(\neq\) value
Yes at \((c,d)\).
Example

\[ f(x,y) = \begin{cases} \frac{x^2}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \]

Continuous? Yes at every \((x,y) \neq (0,0)\) b/c \(\lim_{(x,y) \to (0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0 = f(0,0)\)

partial derivatives

\[ f_x (a,b) = \text{slope} \]

Calculate \(f_x, f_y\) for \(f(x,y) = xy^2 + \sin x\)

\[ f_x = y^2 + \cos x \]

\[ f_y = 2xy \]
\( s_{xy} = (f_x)_y = \text{rate of change of } f_x \text{ in the } y \)-direction.

\[
\begin{align*}
  f_{xx} &= -\sin x \quad f_{yy} = 2x \\
  f_{xy} &= 2y \quad \rightarrow f_{yx} = 2y.
\end{align*}
\]

They are equal! \ldots