Boundary orientation

$S$ a surface oriented by unit normal $\mathbf{n}$, and bounded by a closed curve $C$. Call $C$ the boundary of $S$.

$\partial S = C$ oriented “to the left, standing on $S$ looking at $C$.”
**Example:** S a region in the xy-plane bounded by C. Orienting S by \( n = k \), \( \partial S \) is the usual positive orientation.

**Example:** S part of \( z = 1 - x^2 + y^2 \) with \( z \geq 0 \), oriented with upward pointing normal. Then \( \partial S \) is the unit circle in the xy-plane oriented counterclockwise (positively in the xy-plane).

**Example:** S southern hemisphere in \( x^2 + y^2 + z^2 = 9 \) (i.e. \( z \leq 0 \)), oriented with outward pointing normal on the sphere. \( \partial S \) is the circle of radius 3 in xy-plane \( x^2 + y^2 = 9 \) oriented *clockwise*.
Stokes Theorem

**Theorem.** If $\mathbf{F}$ is a continuously differentiable vector field on $\mathbb{R}^3$, and $S$ is an oriented surface bounded by a simple closed curve, then

$$
\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}
$$
Stokes Theorem: Special case

Suppose \( \mathbf{F}(x, y, z) = \langle P(x, y), Q(x, y), 0 \rangle \), \( S \) a region in the \( xy \)-plane bounded by a simple closed curve, oriented by \( \mathbf{n} = \mathbf{k} \)

\[
\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iiint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}
\]

\[
= \iiint_S \langle 0, 0, Q_x - P_y \rangle \cdot \mathbf{k} \, dS
\]

\[
= \iiint_S Q_x - P_y \, dA \quad \text{Green’s Theorem!!}
\]