Operations on a vector field in $\mathbb{R}^3$: curl

Suppose $\mathbf{F} = \langle P, Q, R \rangle$. Define the curl of $\mathbf{F}$ by

$$\text{curl}(\mathbf{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$  

Write “curl = $\nabla \times$”:

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
P & Q & R
\end{vmatrix}.$$  

Geometric meaning, later: Infinitesimal rotation.
Operations on a vector field in $\mathbb{R}^3$: curl

Example. For $\mathbf{F} = \langle P(x, y), Q(x, y), 0 \rangle$, what is $\text{curl}(\mathbf{F})$?

Example. For a function $f$, what is $\text{curl}(\nabla f)$?

Theorem. Given a vector field $\mathbf{F}$ on $\mathbb{R}^3$ with continuous partial derivatives, $\mathbf{F}$ is conservative if and only if $\text{curl}(\mathbf{F}) = \mathbf{0}$.

Also true for $\mathbf{F}$ on a “simply connected” open subsets $U \subset \mathbb{R}^3$... won’t discuss this as it is more complicated here.
Operations on a vector field in $\mathbb{R}^3$: divergence

Suppose $\mathbf{F} = \langle P, Q, R \rangle$. Define the divergence of $\mathbf{F}$ by

$$\text{div}(\mathbf{F}) = P_x + Q_y + R_z.$$  

Note: this is a function.

Write “$\text{div} = \nabla \cdot$”:

$$\text{div}(\mathbf{F}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z$$

Geometric meaning, later: Expansion of volume.
Example. For a vector field $\mathbf{F}$, what is $\text{div}(\text{curl}(\mathbf{F}))$?

**Theorem.** For any vector field on $\mathbb{R}^3$ with continuous derivatives, $\mathbf{F}$ is the curl of some vector field if and only if $\text{div}(\mathbf{F}) = 0$.

Another important operator on functions:

$$\nabla^2 f = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}.$$

This is called the *Laplacian*, and is often also denoted $\Delta f$. 