Implicit Function Theorem

**Theorem.** Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable and $f(a, b) = 0$.

- If $f_y(a, b) \neq 0$, then there is a differentiable function $g(x)$ defined on an interval $(a - \epsilon, a + \epsilon)$ so that $g(a) = b$ and for every $x \in (a - \epsilon, a + \epsilon)$, we have $f(x, g(x)) = 0$. Furthermore,

  $$g'(x) = -\frac{f_x(x, g(x))}{f_y(x, g(x))} \quad \text{or} \quad \frac{dy}{dx} = -\frac{\partial f}{\partial x} \frac{\partial f}{\partial y}$$

- If $f_x(a, b) \neq 0$, then there is a differentiable function $h(y)$ defined on an interval $(b - \epsilon, b + \epsilon)$ so that $h(b) = a$ and for every $y \in (b - \epsilon, b + \epsilon)$, we have $f(h(y), y) = 0$. Furthermore,

  $$h'(y) = -\frac{f_y(h(y), y)}{f_x(h(y), y)} \quad \text{or} \quad \frac{dx}{dy} = -\frac{\partial f}{\partial y} \frac{\partial f}{\partial x}$$

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Suppose \( f \) is differentiable and \( f(a, b) = 0 \) with \( \nabla f(a, b) \neq 0 \).

Implicit function theorem \( \Rightarrow \)

\[ \nabla f(a, b) \text{ is orthogonal to } f(x, y) = 0 \text{ at } (a, b). \]

Tangent line at \( (a, b) \):

\[ \langle x - a, y - b \rangle \cdot \nabla f(a, b) = 0. \]
**Implicit Function Theorem.** If $f$ is differentiable and $f(a_1, \ldots, a_n) = 0$ with $f_{x_k}(a_1, \ldots, a_n) \neq 0$ some $k = 1, \ldots, n$, then near $(a_1, \ldots, a_n)$, $f(x_1, \ldots, x_n) = 0$ is the graph of a differentiable function of the other $n-1$ variables.

E.g. If $\nabla f(a, b, c) \neq 0$, then near $(a, b, c)$, the level surface $f(x, y, z) = 0$ is the graph of a function of two variables, and the tangent plane is

$$\langle x - a, y - b, z - c \rangle \cdot \nabla f(a, b, c) = 0.$$ 

In general $\nabla f(a_1, \ldots, a_n)$ is orthogonal to “tangent hyperplane” to level set $f(x_1, \ldots, x_n) = 0$. 

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Local max/min

Suppose $\nabla f(a_1, \ldots, a_n) \neq 0$. Then at $(a_1, \ldots, a_n)$
- $f$ increases in direction of $\nabla f(a_1, \ldots, a_n)$ and
- $f$ decreases in direction of $-\nabla f(a_1, \ldots, a_n)$.

**Theorem** At a local max/min, either $f$ fails to be differentiable or $\nabla f = 0$. 

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