For a function $f(x_1, \ldots, x_n)$ and functions $x_1 = x_1(t_1, \ldots, t_k), \ldots, x_n = x_n(t_1, \ldots, t_k)$ the chain rule says:

$$\frac{\partial f}{\partial t_j} = \frac{\partial}{\partial t_j} f(x_1(t_1, \ldots, t_k), \ldots, x_n(t_1, \ldots, t_k)) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t_j}.$$
Directional derivatives

$f : \mathbb{R}^2 \to \mathbb{R}$ and a unit vector $\mathbf{u}$, how does $f$ change in the direction $\mathbf{u}$ at the point $(a, b)$.

Define the *directional derivative of $f$ in the direction $\mathbf{u}$ at the point $(a, b)$* to be

$$D_uf(a, b) = \frac{d}{dt}\bigg|_{t=0} f(a + tu_1, b + tu_2)$$

here $\mathbf{u} = \langle u_1, u_2 \rangle$. (Evaluate the derivative at $t = 0$.)
Example: What are $D_i f(a, b)$ and $D_j f(a, b)$?

In general,

$$D_u f(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2$$

Define the gradient of $f$ at $(a, b)$ as

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle.$$ Then $D_u f(a, b) = \nabla f(a, b) \cdot u$.

Example: Calculate $D_u f(2, -1)$ for $f(x, y) = x^3y + y^2x$ and $u = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$. 
For $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and a unit vector $u = \langle u_1, \ldots, u_n \rangle$ we can similarly define

$$D_uf(a_1, \ldots, a_n) = \frac{d}{dt}igg|_{t=0} f(a_1 + tu_1, \ldots, a_n + tu_n).$$

This is calculated as a dot product with the gradient $\nabla f = \langle f_{x_1}, \ldots, f_{x_n} \rangle$:

$$D_uf(a_1, \ldots, a_n) = \nabla f(a_1, \ldots, a_n) \cdot u$$
Gradient and the chain rule revisited

\( f : \mathbb{R}^n \to \mathbb{R} \) and \( x_1 = x_1(t), \ldots, x_n = x_n(t) \). Write

\[ r(t) = \langle x_1(t), \ldots, x_n(t) \rangle \]

a vector valued function. Set \( r'(t) = \langle x'_1(t), \ldots, x'_n(t) \rangle \)

The chain rule becomes:

\[
\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t).
\]
Meaning of the gradient

Question: For $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$, what is the direction of greatest increase of $f$ at $(a, b)$? I.e. for what unit vector $u$ is $D_u f(a, b)$ largest? And what is this largest value?

Answer:

- Max in direction of $u = \frac{\nabla f(a,b)}{\|\nabla f(a,b)\|}$, and
- Max increase is $\|\nabla f(a, b)\|$.

Assuming $\nabla f(a, b) \neq 0$ (otherwise $D_u f(a, b) = 0$ for all $u$ – why? )