Two theorems on differentiability

**Theorem.** If $f$ is differentiable at $(a, b)$ then $f$ is continuous at $(a, b)$.

**Theorem.** If $f_x$ and $f_y$ exist and are continuous near $(a, b)$ then $f$ is differentiable at $(a, b)$. 
Familiar chain rule for real valued functions of one-variable:

\[
\frac{d}{dx} h(g(x)) = h'(g(x))g'(x).
\]

For functions of several variables, when can we compose?...
Chain rule: version 1.

\[ f : \mathbb{R}^2 \to \mathbb{R} \text{ and } (x, y) : \mathbb{R} \to \mathbb{R}^2 \]

(think: \((x(t), y(t))\) is a parameterized curve).

**Theorem.** If \(f(x, y), x(t), \text{ and } y(t)\) are differentiable, then so is \(f(x(t), y(t))\), and

\[
\frac{df(x(t), y(t))}{dt} = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)
\]

Example: Compute \(h'(\frac{\pi}{4})\), where \(h(t) = f(x(t), y(t))\) and \(f(x, y) = (x + y^2)^2\) and \(x(t) = 2\cos(t), \ y(t) = 2\sin(t)\).
Write $z = f(x, y)$, and $x = x(t)$, $y = y(t)$. Then the chain rule becomes

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.$$
Valid for functions of any number of variables:

\[ f(x_1, \ldots, x_n) \text{ and } x_1 = x_1(t), \ldots, x_n = x_n(t), \text{ then} \]

\[
\frac{df}{dt} = \frac{d}{dt} f(x_1(t), \ldots, x_n(t)) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \frac{dx_k}{dt}.
\]

Example: Calculate \( \frac{dw}{dt} \), where \( w = x^2 + yz \), \( x = t \), \( y = t^2 \), \( z = 1 - t \).
We can also have \( f(x_1, \ldots, x_n) \) and 
\[ x_1 = x_1(t_1, \ldots, t_k), \ldots, x_n = x_n(t_1, \ldots, t_k) \] 
and 
\[
\frac{\partial f}{\partial t_j} = \frac{\partial}{\partial t_j} f(x_1(t_1, \ldots, t_k), \ldots, x_n(t_1, \ldots, t_k)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t_j}.
\]

Since partial derivatives are calculated as ordinary derivatives where we view all but one variable fixed, this is really no different than version 2.