Limit laws

Limits for functions of $n$ variables defined in exactly the same way. Suppose $f, g : \mathbb{R}^n \to \mathbb{R}$ and

$$\lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M.$$ 

($x = (x_1, \ldots, x_n)$ and $a = (a_1, \ldots, a_n)$.) Then

1. $\lim_{x \to a} f(x) + g(x) = L + M$

2. $\lim_{x \to a} f(x)g(x) = LM$

3. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided $M \neq 0$

Example: $\lim_{(x,y,z,w) \to (0,1,1,1)} \frac{x + yz}{w + 1} = \frac{1}{2}$...
A function \( f : \mathbb{R}^n \to \mathbb{R} \) is \textit{continuous at} \( \mathbf{a} = (a_1, \ldots, a_n) \) provided

\[
\lim_{{\mathbf{x} \to \mathbf{a}}} f(\mathbf{x}) = f(\mathbf{a}).
\]

\( f \) is \textit{continuous} if it is continuous at every point \( \mathbf{a} \) in its domain.

Can calculate limit for these functions by “plugging in”... But which functions are continuous?
Some continuous functions

1. Coordinate functions, e.g. \( f(x, y, z) = x, g(x_1, \ldots, x_n) = x_3. \)
2. Sums, differences, products, quotients (where defined):
   e.g. polynomials and rational functions \( f(x, y) = \frac{x+y}{x^2+y^2+1}. \)
3. Compositions of continuous functions: \( f(x, y) = x + y, \)
   \( h(t) = \sin(t), h \circ f(x, y) = \sin(x + y). \)
Partial derivatives

\( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) a function defined at and near \((a, b)\). The \textit{partial derivative with respect to} \(x\) at \((a, b)\) is

\[
f_x(a, b) = \frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}
\]

provided this limit exists.

Think about the function of one variable \(f(x, b)\).

\(f_x(a, b)\) measures the rate of change at \((a, b)\) as we vary \(x\) only.

The partial derivative with respect to \(y\) is similarly defined

\[
f_y(a, b) = \frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}
\]
More on partial derivatives

To calculate, we treat all but the variable in question as fixed.

...Easy!!

Example: \( f(x, y) = xy^2 + \sin(x) \)

Can calculate higher order derivatives as well:

\[
 f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \frac{\partial f}{\partial x}.
\]

or

\[
 f_{xyy} = (f_{xy})_y = \frac{\partial^3 f}{\partial y \partial y \partial x} = \frac{\partial}{\partial y} \frac{\partial^2 f}{\partial y \partial x}.
\]

Note the order of partial derivatives.