Angle and line of intersection for a pair of planes.

There is an example in the notes, but here’s another example finding the line of intersection. Suppose the planes $P_1$ and $P_2$ are given by

$$
\begin{align*}
P_1 : & \quad x - 3y + z = 2 \\
P_2 : & \quad -x + y - 5z = 6
\end{align*}$$

Points on the line are simultaneous solutions to these two equations, and to find the direction vector for the line of intersection, we just need to find two solutions. There are many ways to find solutions to the system. For example, adding the two equations eliminates $x$ and we obtain

$$-2y - 4z = 8$$

or equivalently

$$y + 2z = -4.$$  

Two easy solutions to this are $(y,z) = (0,-2)$ and $(-4,0)$. Using either of the original equations allows us to find the $x$–coordinates corresponding to these and hence two points on the line of intersection $(x,y,z) = (4,0,-2)$ and $(-10,-4,0)$. The displacement vector between these two points is

$$\mathbf{v} = (-10 - 4, -4 - 0, 0 - (-2)) = (-14,-4,2),$$

and this is the direction vector for the line.

To parameterize the line, we use a point on the line, say, $P_0 = (4,0,-2)$ and the direction vector $\mathbf{v} = (-14,-4,2)$. The parametric equations become

$$
(x(t), y(t), z(t)) = (4 + (-14)t, 0 + (-4)t, -2 + 2t)
$$

or equivalently

$$
(x(t), y(t), z(t)) = (4 - 14t, -4t, -2 + 2t).
$$

In vector form, this is

$$\mathbf{r}(t) = \overrightarrow{OP} + t\mathbf{v} = (4,0,-2) + t(-14, -4, 2) = (4 - 14t, -4t, -2 + 2t).$$

This vector form perhaps better explains where the parametric equations come from: every point on the line is obtained by adding a scalar multiple of $\mathbf{v}$ to the direction vector of a point on the line $\overrightarrow{OP}$.

To find the angle, we take the normal vectors to the planes

$$\mathbf{n}_1 = (1,-3,1) \text{ and } \mathbf{n}_2 = (-1,1,-5).$$

The cosine of the angle $\theta$ between the planes is given by

$$\cos(\theta) = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{||\mathbf{n}_1|| ||\mathbf{n}_2||} = \frac{9}{\sqrt{11}(2\sqrt{3})} = \frac{\sqrt{3}}{11}.$$  

and so the angle

$$\theta = \cos^{-1} \left( \frac{\sqrt{3}}{11} \right) \approx 58.5^\circ.$$