Introduction to multiple integrals

1. Evaluate the following integral by reversing the order of integration:

\[ \int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1} \, dx \, dy. \]

(Hint: When you change to \( dx \, dy \), be sure to also change the bounds of integration.)

2. Consider the region bounded by the curves determined by \(-2x + y^2 = 6\) and \(-x + y = -1\).
   
   (a) Sketch the region \( R \) in the plane.
   
   (b) Set up and evaluate an integral of the form \( \iint_R \, dA \) that calculates the area of \( R \).

3. Consider the region \( R \) which lies above the \( x \)-axis and between the circles of radius 1 and 2 centered at (0, 0). Without using polar coordinates, evaluate

\[ \iint_R y \, dA. \]

4. Evaluate

\[ \int_{-2}^0 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx. \]

Hint: don’t do it directly.

5. The function \( P(x) = e^{-x^2} \) is fundamental in probability.

   (a) Sketch the graph of \( P(x) \). Explain why it is called a “bell curve.”

   (b) Compute \( I = \int_{-\infty}^{\infty} e^{-x^2} \, dx \) using the following brilliant strategy of Gauss.

      i. Instead of computing \( I \), compute \( I^2 = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2} \, dy \right). \)

      ii. Rewrite \( I^2 \) as an integral of the form \( \iint_R f(x, y) \, dA \) where \( R \) is the entire Cartesian plane.

      iii. Convert that integral to polar coordinates.

      iv. Evaluate to find \( I^2 \). Deduce the value of \( I \).

Amazingly, it can be mathematically proven that there is NO elementary function \( Q(x) \) (that is, function built up from sines, cosines, exponentials, and roots using “usual” operations) for which \( Q'(x) = P(x) \).

6. Compute

\[ \int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2} \, dx \, dy. \]