Thursday, January 31  *  Solutions  *  Functions of several variables; Limits.

1. For each of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$, draw a sketch of the graph together with pictures of some level sets.

(a) $f(x, y) = xy$

(b) $f(x) = |x|$. Please note here that $x$ is a vector. In coordinates, this function is $f(x, y) = \sqrt{x^2 + y^2}$. 

For (a), the result is one of the many quadric surfaces. What is the name for this type? Is the graph in (b) also a quadric surface?

Solution.

(a) The graph of the function $f(x, y) = xy$ is

![Graph of f(x, y) = xy](image1.png)

Figure 1: Graph of $f(x, y) = xy$.

The graph of the level sets $f(x, y) = -2, -1, 0, 1, 2$ is

![Graph of Level Sets of f(x, y) = xy](image2.png)

Figure 2: Graph of Level Sets of $f(x, y) = xy$. 

The graph of \( f(x, y) = xy \) is a hyperbolic paraboloid since the horizontal traces are hyperbolas and the vertical traces are parabolas.

(b) The graph of the function \( f(x) = |x| \) is

The graph of the level sets \( f(x, y) = 0, 1, 2, 3 \) is

The graph of \( f(x) = |x| \) is not a quadric surface because it cannot be written as \( Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0 \). It is the top half of a cone, which is a quadric surface.

2. Consider the function \( f : \mathbb{R}^2 \to \mathbb{R} \) given by

\[
f(x, y) = \frac{2x^3 y}{x^6 + y^2} \quad \text{for } (x, y) \neq 0
\]

In this problem, you'll consider \( \lim_{(x, y) \to 0} f(x, y) \).
(a) Look at the values of \( f \) on the \( x \)- and \( y \)-axes. What do these values show the limit \( \lim_{(x,y) \to 0} f(x, y) \) must be if it exists?

**Solution.** Along \( y = 0 \), \( \lim_{(x,y) \to 0} f(x, y) = \lim_{x \to 0} f(x,0) = \lim_{x \to 0} 0 = 0. \)

Along \( x = 0 \), \( \lim_{(x,y) \to 0} f(x, y) = \lim_{y \to 0} f(0,y) = \lim_{y \to 0} 0 = 0. \)

Thus, should it exist, we must have \( \lim_{(x,y) \to 0} f(x, y) = 0. \)

(b) Show that along each line in \( \mathbb{R}^2 \) through the origin, the limit of \( f \) exists and is 0.

**Solution.** Any line through the origin besides \( x = 0 \) or \( y = 0 \) can be written as \( y = mx, \ m \neq 0. \)

Along \( y = mx \), \( \lim_{(x,y) \to 0} f(x, y) = \lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{2mx^4}{x^6 + m^2x^2} = \lim_{x \to 0} \frac{2mx^2}{x^4 + m^2} = 0. \)

(c) Despite this, show that the limit \( \lim_{(x,y) \to 0} f(x, y) \) does not exist by finding a curve over which \( f \) takes on the constant value 1.

**Solution.** Along \( y = x^3 \), \( \lim_{(x,y) \to 0} f(x, y) = \lim_{x \to 0} f(x, x^3) = \lim_{x \to 0} \frac{2x^6}{x^6 + x^6} = 1. \)

3. Consider the function \( f : \mathbb{R}^2 \to \mathbb{R} \) given by

\[
f(x, y) = \frac{xy^2}{\sqrt{x^2 + y^2}} \quad \text{for} \ (x, y) \neq 0
\]

In this problem, you'll show \( \lim_{h \to 0} f(h) = 0. \)

(a) For \( \epsilon = 1/2 \), find some \( \delta > 0 \) so that when \( 0 < |h| < \delta \) we have \( |f(h)| < \epsilon \). Hint: As with the example in class, the key is to relate \( |x| \) and \( |y| \) with \( |h| \).

**Solution.** Note that \( |x|, |y| \leq |h| \). For \( \epsilon = 1/2 \), let \( \delta = 1/\sqrt{2} \). Then \( 0 < |h| < \delta \) implies

\[
|f(h)| \leq \frac{|h|^3}{|h|} = |h|^2 < \delta^2 = \frac{1}{2}.
\]

(b) Repeat with \( \epsilon = 1/10. \)

**Solution.** For \( \epsilon = 1/10 \), let \( \delta = 1/\sqrt{10} \). Then \( 0 < |h| < \delta \) implies

\[
|f(h)| \leq |h|^2 < \delta^2 = \frac{1}{10}.
\]
(c) Now show that \( \lim_{h \to 0} f(h) = 0 \). That is, given an arbitrary \( \epsilon > 0 \), find a \( \delta > 0 \) so that when \( 0 < |h| < \delta \) we have \( |f(h)| < \epsilon \).

**Solution.** Given \( \epsilon > 0 \), let \( \delta = \sqrt{\epsilon} \). Then \( 0 < |h| < \delta \) implies

\[
|f(h)| \leq |h|^2 < \delta^2 = \epsilon.
\]

(d) Explain why the limit laws that you learned in class on Wednesday aren't enough to compute this particular limit.

**Solution.** \( f(x, y) \) cannot be written as \( f(x, y) = g(x, y)h(x, y) \) so that \( \lim_{|x| \to 0} g(x) \) and \( \lim_{|x| \to 0} h(x) \) both exist and are easier to compute than \( \lim_{|x| \to 0} f(x) \).