Thursday, January 31  

** Functions of several variables; Limits. 

1. For each of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$, draw a sketch of the graph together with pictures of some level sets.

   (a) $f(x, y) = xy$
   (b) $f(x) = |x|$. Please note here that $x$ is a vector. In coordinates, this function is $f(x, y) = \sqrt{x^2 + y^2}$.

   For (a), the result is one of the many quadric surfaces. What is the name for this type? Is the graph in (b) also a quadric surface?

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

   $$f(x, y) = \frac{2x^3y}{x^6 + y^2} \quad \text{for} \ (x, y) \neq 0$$

   In this problem, you’ll consider $\lim_{(x,y)\to0} f(x, y)$.

   (a) Look at the values of $f$ on the $x$- and $y$-axes. What do these values show the limit $\lim_{(x,y)\to0} f(x, y)$ must be if it exists?

   (b) Show that along each line in $\mathbb{R}^2$ through the origin, the limit of $f$ exists and is 0.

   (c) Despite this, show that the limit $\lim_{(x,y)\to0} f(x, y)$ does not exist by finding a curve over which $f$ takes on the constant value 1.

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

   $$f(x, y) = \frac{xy^2}{\sqrt{x^2 + y^2}} \quad \text{for} \ (x, y) \neq 0$$

   In this problem, you’ll show $\lim_{h\to0} f(h) = 0$.

   (a) For $\epsilon = 1/2$, find some $\delta > 0$ so that when $0 < |h| < \delta$ we have $|f(h)| < \epsilon$. Hint: As with the example in class, the key is to relate $|x|$ and $|y|$ with $|h|$.

   (b) Repeat with $\epsilon = 1/10$.

   (c) Now show that $\lim_{h\to0} f(h) = 0$. That is, given an arbitrary $\epsilon > 0$, find a $\delta > 0$ so that that when $0 < |h| < \delta$ we have $|f(h)| < \epsilon$.

   (d) Explain why the limit laws that you learned in class on Wednesday aren’t enough to compute this particular limit.