Tuesday, January 15  *  Solutions  *  A review of some important calculus topics

1. Chain Rule:

   (a) Let \( h(t) = \sin(\cos(\tan t)) \). Find the derivative with respect to \( t \).

   **Solution.**

   \[
   \frac{d}{dt}(h(t)) = \frac{d}{dt}(\sin(\cos(\tan t)))
   = \cos(\cos(\tan t)) \cdot \frac{d}{dt}(\cos(\tan t))
   = \cos(\cos(\tan t)) \cdot (\cos(\tan t)) \cdot \frac{d}{dt}(\cos(\tan t))
   = \cos(\cos(\tan t)) \cdot (-\sin(\tan t)) \cdot \sec^2 t
   \]

   (b) Let \( s(x) = \sqrt{x} \) where \( x(t) = \ln(f(t)) \) and \( f(t) \) is a differentiable function. Find \( \frac{ds}{dt} \).

   **Solution.** From \( \frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt} \), we get

   \[
   \frac{ds}{dt} = \frac{1}{4x^{3/4}} \cdot f'(t) \cdot f(t).
   \]

   But we need to make sure that \( \frac{ds}{dt} \) is a single variable function of \( f \), so

   \[
   \frac{ds}{dt} = \frac{1}{4[\ln(f(t))]^{3/4}} \cdot f'(t) \cdot f(t).
   \]

2. Parameterized curves:

   (a) Describe and sketch the curve given parametrically by

   \[
   \begin{cases}
   x = 5 \sin(3t) \\
   y = 3 \cos(3t)
   \end{cases}
   \text{for } 0 \leq t < \frac{2\pi}{3}.
   \]

   What happens if we instead allow \( t \) to vary between 0 and \( 2\pi \)?

   **Solution.** Note that

   \[
   \left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = \sin^2(3t) + \cos^2(3t) = 1.
   \]

   So this parameterizes (at least part of) the ellipse \( \left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \).
By examining differing values of $t$ in $0 \leq t \leq \frac{2\pi}{3}$, we see that this parametrization travels the ellipse in a clockwise fashion exactly once.

- $t = 0: (x(0), y(0)) = (0, 3)$
- $t = \pi/6: (x(\pi/6), y(\pi/6)) = (5, 0)$
- $t = \pi/3: (x(\pi/3), y(\pi/3)) = (0, -3)$
- $t = \pi/2: (x(\pi/2), y(\pi/2)) = (-5, 0)$

![Figure 1: Ellipse.](image)

If we let $t$ vary between 0 and $2\pi$, we will traverse the ellipse 3 times.

(b) Set up, but **do not evaluate** an integral that calculates the arc length of the curve described in part (a).

**Solution.** Arc length

\[
s = \int_{a}^{b} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt
\]

\[
= \int_{0}^{\frac{2\pi}{3}} \sqrt{(15 \cos(3t))^2 + (-9 \sin(3t))^2} \, dt.
\]

(c) Consider the equation $x^2 + y^2 = 16$. Graph the set of solutions of this equation in $\mathbb{R}^2$ and find a parametrization that traverses the curve once counterclockwise.

**Solution.** If we let $x = 4 \cos t$ and $y = 4 \sin t$, then $x^2 + y^2 = (4 \cos t)^2 + (4 \sin t)^2 = 16$. More-
over, as \( t \) increases, this parametrization traverses the circle in a counterclockwise fashion:

\[
\begin{align*}
t &= 0 : (x(0), y(0)) = (4, 0) \\
t &= \pi/2 : (x(\pi/2), y(\pi/2)) = (0, 4) \\
t &= \pi : (x(\pi), y(\pi)) = (-4, 0) \\
t &= 3\pi/2 : (x(3\pi/2), y(3\pi/2)) = (0, -4) \\
t &= 2\pi : (x(2\pi), y(2\pi)) = (4, 0)
\end{align*}
\]

To ensure that we travel the curve only once, we restrict \( t \) to the interval \([0, 2\pi)\). So the parametrization is

\[
\begin{cases}
x = 4 \cos t \\ y = 4 \sin t
\end{cases}
\quad \text{when} \quad 0 \leq t \leq 2\pi.
\]

3. 1st and 2nd Derivative Tests:

(a) Use the 2nd Derivative Test to classify the critical numbers of the function \( f(x) = x^4 - 8x^2 + 10 \).

**Solution.** First, we find the critical points of \( f(x) \).

\[
f'(x) = 4x^3 - 16x.
\]

\( f'(x) = 0 \) when \( 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2) = 0 \). Hence \( f''(x) = 0 \) when \( x = 0, x = 2 \) or \( x = -2 \).
Now apply the 2nd Derivative Test to the three critical points. From \( f''(x) = 12x^2 - 16 \), we get:

\( f''(0) = -16 < 0 \), so \( y = f(x) \) is concave down at the point \((0, f(0))\). So a local max occurs at \((0, 10)\).

\( f''(-2) = 32 > 0 \), so \( y = f(x) \) is concave up at the point \((-2, f(-2))\). So a local min occurs at \((-2, -6)\).

\( f''(2) = 32 > 0 \), so \( y = f(x) \) is concave up at the point \((2, f(2))\). So a local min occurs at \((2, -6)\).

\( b) \) Use the 1st Derivative Test and find the extrema of \( h(s) = s^4 + 4s^3 - 1 \).

**Solution.** First, find the critical points of \( h(s) \).

\[ h'(s) = 4s^3 + 12s^2. \]

Then \( h'(s) = 0 \) when \( 4s^3 + 12s^2 = 4s^2(s + 3) = 0 \). So \( h'(s) = 0 \) when \( s = 0 \) and \( s = -3 \).

For the 1st Derivative Test, we need to determine if \( h \) is increasing or decreasing on the intervals \((-\infty, -3), (-3, 0) \) and \((0, \infty)\).

On \((-\infty, -3)\) choose any test point (for example, choose \( s = -1000 \)). The sign of \( h'(s) = 4s^3 + 12s^2 < 0 \) on this interval. Hence \( h(s) \) is decreasing on \((-\infty, -3)\).

On \((-3, 0)\) choose any test point (for example, choose \( s = -1 \)). The sign of \( h'(s) = 4s^3 + 12s^2 > 0 \) on this interval. Hence \( h(s) \) is increasing on \((-3, 0)\).

On \((0, \infty)\) choose any test point (for example, choose \( s = 1000 \)). The sign of \( h'(s) = 4s^3 + 12s^2 > 0 \) on this interval. Hence \( h(s) \) is increasing on \((0, \infty)\).

Since at \( s = -3 \) the function changes from decreasing to increasing, the function must have obtained a local min at \( s = -3 \).

At \( s = 0 \), neither a max or a min occurs in the value of \( h \).

\( c) \) Explain why the 2nd Derivative test is unable to classify all the critical numbers of \( h(s) = s^4 + 4s^3 - 1 \).

**Solution.** When \( s = -3 \), \( h''(-3) = 36 > 0 \). A local min occurs when \( s = -3 \) by the 2nd Derivative Test.

When \( s = 0 \), \( h''(0) = 0 \). The 2nd Derivative Test is inconclusive. The graph of \( y = h(s) \) has no concavity at \((0, h(0))\). Without more information (the 1st Derivative Test), we are unable to identify \((0, h(0))\) as a local max, min or a point of inflection.

4. Consider the function \( f(x) = x^2 e^{-x} \).
(a) Find the best linear approximation to $f$ at $x = 0$.

**Solution.** Recall that in Calc I and II, the "best linear approximation" is synonymous with the equation of the tangent line or the 1st order Taylor polynomial. Hence, $f'(x) = 2xe^{-x} + x^2(-e^{-x})$.

Since $f'(0) = 0$, the tangent line has no slope at $(0, f(0)) = (0, 0)$. The equation of the tangent line is $y = 0$.

(b) Compute the second-order Taylor polynomial at $x = 0$.

**Solution.** By definition, the second-order Taylor polynomial at $x = 0$ is

$$T_2(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2.$$  

Since $f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x}$, we compute that $f''(0) = 2$. Hence

$$T_2(x) = 0 + \frac{0}{1!}(x-0) + \frac{2}{2!}(x-0)^2 = x^2.$$  

(c) Explain how the second-order Taylor polynomial at $x = 0$ demonstrates that $f$ must have a local minimum at $x = 0$.

**Solution.** The second-order Taylor polynomial is the best quadratic approximation to the curve $y = f(x)$ at the point $(0, f(0))$. Since $T_2(x) = x^2$ clearly has a local minimum at $(0, 0)$, and $(0, 0)$ is the location of a critical point of $f$, then $f$ must also have a local minimum at $(0, 0)$.

5. Consider the integral $\int_0^{\sqrt{3\pi}} 2x \cos(x^2) \, dx$.

(a) Sketch the area in the $xy$-plane that is implicitly defined by this integral.

**Solution.** The shadow area in the following picture is the area defined by the integral.
(b) To evaluate, you will need to perform a substitution. Choose a proper \( u = f(x) \) and rewrite the integral in terms of \( u \). Sketch the area in the \( uv \)-plane that is implicitly defined by this integral.

**Solution.** Let \( u = x^2 \). Then \( du = 2x \, dx \), so the integral becomes

\[
\int_{0}^{\sqrt{3\pi}} 2x \cos(x^2) \, dx = \int_{0}^{3\pi} \cos u \, du.
\]

(c) Evaluate the integral \( \int_{0}^{\sqrt{3\pi}} 2x \cos(x^2) \, dx \).

**Solution.**

\[
\int_{0}^{\sqrt{3\pi}} 2x \cos(x^2) \, dx = \int_{0}^{3\pi} \cos u \, du = \left[ \sin u \right]_{u=0}^{u=3\pi} = \sin(3\pi) - \sin(0) = 0.
\]