Test 2(b), Math 241,  
Tuesday March 12, 2013

Name and Section ________________________________

Instructions Show all work: no work = no partial credit. On many problems you can check your work, do so if possible. The test is worth 100 points.

1
2
3
4
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6
7
1. (20 points) Suppose $f$ is a function of two variables with continuous partial derivatives of all orders, and that it has values and partial derivatives as in the table at the right.

(a) (10 points) Find all the critical points you can from the given data and classify them into local mins, local maxes, and saddles.

\[
\begin{array}{c}
(0,0): \\
\mathbf{D} = 2(3) - 3^2 = 6 - 9 = -3 < 0 \\
\mathbf{D} = (-4)x^2 - 2y^2 = 8 - 4 = 4 > 0 \\
\mathbf{f}_{xx}(0,0) = -4 < 0 \\
\end{array}
\]

\[
\begin{array}{c}
(1,0): \\
\mathbf{D} = 3(1) - 0 = 3 > 0 \\
\mathbf{f}_{xx}(1,0) = 3 > 0 \\
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
(x,y) & f & f_x & f_y & f_{xx} & f_{yy} & f_{xy} \\
\hline
(0,0) & 1 & 0 & 0 & 2 & 3 & 3 \\
(0,1) & 4 & 0 & 1 & 1 & 2 & -1 \\
(-1,0) & -1 & 0 & 0 & -4 & -2 & 2 \\
(1,1) & 3 & 0 & 0 & 3 & 1 & 0 \\
(1,-1) & -2 & 1 & 0 & 2 & 4 & -3 \\
\hline
\end{array}
\]

(b) (5 points) Find the equation of the tangent line to the level curve $f(x, y) = -2$ at the point $(1, -1)$.

\[
\nabla f(x, y) \cdot (x-1, y+1) = 0 \\
(1, -1) \cdot (x-1, y+1) = 0 \\
x - 1 = 0
\]

\[
\frac{x}{1} + \frac{y}{1} = 1
\]

(c) (5 points) Calculate the directional derivative $D_v f(0,1)$, where $v = \frac{\sqrt{3}}{2} i + \frac{1}{2} j$.

\[
D_v f(0,1) = \nabla f(0,1) \cdot \frac{\vec{v}}{
abla f(0,1) = \left(0, 1\right) \cdot \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \\
= \frac{1}{2}
\]
2. (10 points) Let \( f(x, y) = xy \). Find max and min values of \( f \), subject to the constraint \( x^2 + y^2 = 4 \), as follows:

(a) (2 points) Set up the problem using Lagrange multipliers.
\[
\begin{align*}
q(x, y) &= x^2 + y^2 = 4 \\
\nabla q &= (2x, 2y) \\
\nabla f &= (y, x)
\end{align*}
\]
\[
\begin{cases}
\nabla f = \lambda \nabla q \\
q = 4
\end{cases}
\Rightarrow \begin{cases}
y = 2\lambda x \\
x = 2\lambda y \\
x^2 + y^2 = 4
\end{cases}
\]

(b) (5 points) Solve for \( \lambda, x, y \).
\[
\begin{align*}
\cdot x = 0 &\Rightarrow y = 0 \Rightarrow 0^2 + 0^2 + 4 \neq 4 \times \text{X} \\
\cdot x \neq 0 &\Rightarrow x = \pm \sqrt{4} \Rightarrow x^2 = 4 \Rightarrow \lambda = \pm \frac{1}{2}
\end{align*}
\]
\[
\begin{align*}
\Rightarrow y = \pm x \\
\Rightarrow x^2 + y^2 = 4 \\
x = \pm \sqrt{2}
\end{align*}
\]

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( (x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\lambda}{2} )</td>
<td>( (\sqrt{2}, 2\sqrt{2}) )</td>
</tr>
<tr>
<td>( -\frac{\lambda}{2} )</td>
<td>( (\sqrt{2}, -2\sqrt{2}) )</td>
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<tr>
<td>( \frac{\lambda}{2} )</td>
<td>( (-\sqrt{2}, \sqrt{2}) )</td>
</tr>
<tr>
<td>( -\frac{\lambda}{2} )</td>
<td>( (-\sqrt{2}, -\sqrt{2}) )</td>
</tr>
</tbody>
</table>

(c) (3 points) Use your answer from part (b) to find the max and min values of \( f \). Check if possible.
\[
\begin{align*}
\Rightarrow f(\sqrt{2}, \sqrt{2}) = f(-\sqrt{2}, -\sqrt{2}) &= 2 \\
\Rightarrow f(\sqrt{2}, -\sqrt{2}) = f(-\sqrt{2}, \sqrt{2}) &= -2
\end{align*}
\]
\[
\text{Max} = 2, \ \text{Min} = -2
\]
3. (15 points) Consider the force field \( \mathbf{F} = 2xi - 2yj \) acting on an object whose trajectory traces out the arc of the parabola \( y = 2 - x^2 \) from \((-1, 1)\) to \((0, 2)\).

(a) (3 points) Find a parameterization of the path of the object.

\[
\mathbf{r}(t) = \begin{pmatrix} t \\ 2-t^2 \end{pmatrix}, \quad \text{with} \quad -1 \leq t \leq 0
\]

(b) (5 points) Calculate directly the work done by \( \mathbf{F} \) on the object.

\[
\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{0} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_{-1}^{0} (2t, -2(t^2-2)) \cdot (1, -2t) \, dt
\]

\[
= \int_{-1}^{0} (2t^2 - 4t^3) \, dt = \left. \left( \frac{2t^3}{3} - t^4 \right) \right|_{-1}^{0} = -(5-1) = -4
\]

Work = \[ -4 \]

(c) (5 points) The force field \( \mathbf{F} \) is conservative. Find a potential function \( f \) so that \( \nabla f = \mathbf{F} \).

\[
f(x, y) = x^2 - y^2
\]

Check: \( \nabla f = (2x, -2y) = \mathbf{F} \).

\[
f(x, y) = \begin{pmatrix} x^2 - y^2 \end{pmatrix}
\]

(d) (2 points) Use your answer from part (c) to check your answer from part (b). If you couldn’t find \( f \) in part (c), explain how you would use such an \( f \) to check your answer.

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(0, 2) - f(1, 1) = (0 - 4) - (1 - 1) = -4 \quad \checkmark
\]
4. (15 points) Consider the curve $C$ which admits a parameterization as $\mathbf{r}(t) = (-e^t, e^{-t})$ for $-1 \leq t \leq 1$.

(a) (5 points) Circle the picture that most accurately illustrates the curve $\mathbf{r}(t)$?

(b) (5 points) The length of $C$ is $L \approx 3.2557$. Circle the integral that calculates $L$.

\[
\int_{-e}^{e} \sqrt{1 + e^{2t}} \, dt \quad \int_{-e}^{e} \sqrt{e^{2t} + e^{-2t}} \, dt \quad \int_{-1}^{1} \sqrt{e^t + e^{-t}} \, dt \quad \int_{-1}^{1} \sqrt{e^{2t} + e^{-2t}} \, dt.
\]

(c) (5 points) Circle the integral representing $\int_{C} x^2y \, ds$.

\[
\int_{-e}^{e} e^{2t} \sqrt{e^{2t} + e^{-2t}} \, dt \quad \int_{-e}^{e} e^t \sqrt{1 + e^{2t}} \, dt \quad \frac{1}{L} \int_{-e}^{e} e^{2t} \sqrt{e^{2t} + e^{-2t}} \, dt \quad \frac{1}{L} \int_{-e}^{e} e^t \sqrt{1 + e^{2t}} \, dt
\]

\[
\int_{-1}^{1} e^{2t} \sqrt{e^t + e^{-t}} \, dt \quad \int_{-1}^{1} e^t \sqrt{e^{2t} + e^{-2t}} \, dt \quad \frac{1}{L} \int_{-1}^{1} e^{2t} \sqrt{e^t + e^{-t}} \, dt \quad \frac{1}{L} \int_{-1}^{1} e^t \sqrt{e^{2t} + e^{-2t}} \, dt
\]
5. (15 points) Consider the three vector fields $\mathbf{E}, \mathbf{G}, \mathbf{H}$ shown.

(a) (5 points) Circle the letter of the vector field which is $\frac{-y}{10}\mathbf{i} + \frac{x}{10}\mathbf{j}$.  

(b) (5 points) Exactly one of the vector fields $\mathbf{E}, \mathbf{G}, \mathbf{H}$ is conservative. Circle the letter of the vector field which is conservative.

(c) (5 points) If $C$ is the straight line segment from $(1,0)$ to $(0,1)$, circle the letter(s) of the vector field(s) $\mathbf{F}$ for which $\int_C \mathbf{F} \cdot d\mathbf{r}$ positive.
6. (10 points) Consider the curve $C$ in $\mathbb{R}^3$ that has a parameterization

$$\mathbf{r}(t) = (\cos(\pi t), t, \sin(\pi t))$$

with $0 \leq t \leq 3$.

(a) (5 points). Circle the letter of the picture that most accurately illustrates the curve.

(b) (5 points). Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ where $f(x, y, z) = x^2 + y^2 + z^2$.

$$\int_C \nabla f \cdot d\mathbf{r} = \int_{\mathbf{r}(1)}^{\mathbf{r}(3)} \left( \nabla f \right) \cdot d\mathbf{r} = f(3, 3, 0) - f(-1, 0, 0) = 27 - 1 = 9$$

$$\int_C \nabla f \cdot d\mathbf{r} = 9$$
7. (15 points) Suppose \( F: D \rightarrow \mathbb{R}^2 \) is a vector field written as \( F = Pi + Qj \) where \( P(x,y) \) and \( Q(x,y) \) are continuously differentiable functions on the connected open set \( D \). Suppose that

\[
\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}
\]

and that there is a closed curve \( C \) contained in \( D \) so that

\[
\int_C F \cdot dr = -7.
\]

Decide whether each of the following statements is true or false and then circle your answer.

(a) \( F \) has the path independence property. 

(b) \( D \) is simply connected.

(c) \( \int_C F \cdot dr = 7 \).

(d) For every closed curve \( C_0 \) in \( D \) we have \( \int_{C_0} F \cdot dr = -7 \).

(e) There is some closed curve \( C_0 \) in \( D \) so that \( \int_{C_0} F \cdot dr = 0 \).