Test 1(a), Math 241,
Tuesday Feb 12, 2013

Name and Section __________________________________________

Instructions Show all work: no work = no partial credit. On many problems you can check your work, do so if possible. The test is worth 100 points.

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1. (10 points)

(a) (5 points) Find an equation in \((x, y, z)\) of the plane thru the points \((1, 1, 1)\), \((3, 2, 1)\), \((1, 2, 0)\).

\[
\overrightarrow{PQ} = (2, 1, 0), \quad \overrightarrow{PR} = (0, 1, -1)
\]

\[
\hat{n} = \begin{vmatrix}
2 & 1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 1
\end{vmatrix} = (-1, 2, 1)
\]

\[-1(x-1) + 2(y-1) + z(z-1) = 0
\]

\[-x + 2y + 2z + 1 - 2 - z = 0
\]

\[-x + 2y + 2z = 3
\]

\[
-1x + 2y + 2z = 3
\]

(b) (5 points) Find the distance from the plane \(x + y + z = 1\) to the point \((0, 1, 7)\).

\[P = (0, 1, 0) \text{ on plane } (0 + 1 + 0 = 1)
\]

\[\overrightarrow{PQ} = (0, 0, 7), \quad \hat{n} = (1, 1, 1) \text{ normal to plane}
\]

\[
dist = |\text{proj}_n(\overrightarrow{PQ})| = \frac{|\overrightarrow{PQ} \cdot \hat{n}|}{|\hat{n}|} = \frac{7}{\sqrt{3}}
\]

\[\text{distance} = \frac{7}{\sqrt{3}}
\]
2. (5 points) Find $\text{proj}_v w$ if $v = (1, 1, 1)$ and $w = (1, 1, 2)$.

$$\text{proj}_v w = \frac{\overrightarrow{v} \cdot \overrightarrow{w}}{|\overrightarrow{v}|^2} \overrightarrow{v} = \frac{1 + 1 + 2}{3} (1, 1, 1) = \left( \frac{4}{3}, \frac{4}{3}, \frac{4}{3} \right)$$

$$\text{proj}_v w = \left( \frac{4}{3}, \frac{4}{3}, \frac{4}{3} \right)$$

3. (5 points) If $a$ and $b$ are any two vectors in $\mathbb{R}^3$ then (circle one)

(A.) $|a \times b|^2 + (a \cdot b)^2 = |a|^2 |b|^2$

$$|a \times b|^2 + (a \cdot b)^2 = (|a||b| \sin \theta)^2 + (|a||b| \cos \theta)^2 = |a|^2 |b|^2 (\sin^2 \theta + \cos^2 \theta) = |a|^2 |b|^2$$

(B.) $|a \times b|^2 + (a \cdot b)^2 \neq |a|^2 |b|^2$

(C.). Neither (A.) or (B.); the answer depends on the choice of the vectors.

4. (5 points) Find the cosine of the angle $\theta$ between the two planes defined by equations $3x + 2y + z = 1$ and $2x - y + z = 4$.

$$\vec{n}_1 = (3, 2, 1), \quad \vec{n}_2 = (2, -1, 1)$$

$$\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{6 - 2 + 1}{\sqrt{9 + 4 + 1} \sqrt{4 + 1 + 1}} = \frac{5}{\sqrt{14} \sqrt{6}} = \frac{5}{2 \sqrt{21}}$$

$$\cos(\theta) = \frac{5}{2 \sqrt{21}}$$
5. (15 points) For vectors \( v = (0, 1, 3), u = (3, 4, 2), w = (1, 0, 1) \), compute

(a.) (4 points) \( v \times u \)
\[
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & 1 & 3 \\
3 & 4 & 2 \\
\end{vmatrix} = (2-12) - (-9), -3)
\]
\[v \times u = (-10, 9, -3)\]

(b.) (4 points) \( w \cdot (v \times u) \)
\[
(1, 0, 1) \cdot (-10, 9, -3) = -10 - 3
\]
\[w \cdot (v \times u) = -13\]

(c.) (4 points) Cosine of the angle \( \theta \) between \( w \) and \( (v \times u) \)
\[
\frac{\vec{w} \cdot (\vec{v} \times \vec{u})}{\|\vec{w}\| \|\vec{v} \times \vec{u}\|} = \frac{-13}{\sqrt{2} \sqrt{100 + 81 + 9}} = \frac{-13}{\sqrt{2}190}
\]
\[
\cos(\theta) = \frac{-13}{2\sqrt{95}}
\]

(d.) (3 points) Volume of the parallelepiped determined by \( u, v, w \).
\[
|\vec{w} \cdot (\vec{v} \times \vec{u})|
\]
\[Volume = 13\]
6. (10 points) Sketch the five level sets \( f(x, y) = -2, -1, 0, 1, 2 \) for the function

\[
f(x, y) = (x - 1)^2 - (y + 2)^2
\]

and label each level set with the corresponding value \(-2, -1, 0, 1, 2\).
7. (10 points) Calculate the partial derivatives \( f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx} \) at \((0, 0)\), then decide which of the pictures most accurately illustrates the part of the graph near \((0, 0)\).

\[
\begin{align*}
f(x, y) &= \sin(3x^2 + 4y^2) \\
f_x &= \cos(3x^2 + 4y^2)(6x) \\
f_y &= \cos(3x^2 + 4y^2)(8y) \\
f_{xx} &= -\sin(3x^2 + 4y^2)(6x)^2 + \cos(3x^2 + 4y^2)(6) \\
f_{yy} &= -\sin(3x^2 + 4y^2)(8y)^2 + \cos(3x^2 + 4y^2)(8) \\
f_{xy} &= f_{yx} = 0
\end{align*}
\]

\[
\begin{align*}
f_x(0, 0) &= 0 \\
f_y(0, 0) &= 0 \\
f_{xx}(0, 0) &= 6 \\
f_{yy}(0, 0) &= 8 \\
f_{xy}(0, 0) &= 0 \\
f_{yx}(0, 0) &= 0
\end{align*}
\]

Graph: **A**
8. (10 points) Determine if the limit exists. If so, say what the limit is and justify your answer. If not, show that limits along two different paths are not equal.

(a.) (5 points) \( \lim_{(x,y) \to (0,0)} \frac{x^2 - 3y^2}{x^2 + 17y^2 + 1} \)

RATIONAL FUNCTIONS ARE CONTINUOUS WHEN DENOMINATOR IS NONZERO. \( x^2 + 17y^2 + 1 \) IS NEVER 0, SO \( \frac{x^2 - 3y^2}{x^2 + 17y^2 + 1} \) IS CONTINUOUS EVERYWHERE, SO

\[
\lim_{(x,y) \to (0,0)} \frac{x^2 - 3y^2}{x^2 + 17y^2 + 1} = \frac{0 - 0}{0 + 0 + 1} = 0
\]

\[
\lim_{(x,y) \to (0,0)} \frac{x^2 - 3y^2}{x^2 + 17y^2 + 1} = 0
\]

(b.) (5 points) \( \lim_{(x,y) \to (0,0)} \frac{\sin^2 x}{x^2 + y^2} \)

\( x = 0 : \quad \frac{y^7 \cdot 0}{0 + y^5} = 0 \to 0 \) as \( y \to 0 \)

\( y = x : \quad \frac{x^5 \sin^2 x}{x^5 + x^5} = \frac{x^3 \sin^2 x}{2x^2} = \frac{1}{2} \left( \frac{\sin x}{x} \right)^2 \to \frac{1}{2} \cdot 1 \) \hspace{1cm} \text{as} \ x \to 0

\[
\frac{1}{2} \neq 0
\]

So limit does not exist.

\[
\lim_{(x,y) \to (0,0)} \frac{\sin^2 x}{x^2 + y^2} = \boxed{\text{DNE}}
\]
9. (10 points)

(a.) (4 points) Explain why the function \( z = \frac{y}{y-x} \) is differentiable at \((3,1)\) by evaluating the appropriate limit or quoting a theorem.

\[
\frac{\partial z}{\partial x} = \frac{(y-x)(-1) - y(-1)}{(y-x)^2} = \frac{-1}{(y-x)^2} \quad \text{BECAUSE THESE FUNCTIONS ARE RATIONAL,}
\]

\[
\frac{\partial z}{\partial y} = \frac{(y-x)(1) - y(1)}{(y-x)^2} = \frac{-1}{(y-x)^2} \quad \text{HENCE CONTINUOUS NEAR \((3,1)\), THE}
\]

\[
\frac{\partial z}{\partial y} = \frac{y}{(y-x)^2} \quad \text{FUNCTION IS DIFFERENTIABLE AT \((3,1)\).}
\]

(b.) (6 points) Find an equation in \((x, y, z)\) for the tangent plane to \( z = \frac{y}{y-x} \) at the point \((3,1)\)

\[
\frac{\partial z}{\partial x}(3,1) = \frac{1}{(1-3)^2} = \frac{1}{4} \\
\frac{\partial z}{\partial y}(3,1) = \frac{-3}{(1-3)^2} = -\frac{3}{4}
\]

\[
L(x, y) = \frac{1}{1-3} + \frac{1}{4}(x-3) + \left(\frac{-3}{4}\right)(y-1)
\]

\[
= \frac{1}{2} + \frac{1}{4}x - \frac{3}{4}y - \frac{3}{4} + \frac{3}{4}
\]

\[
\Rightarrow \quad \frac{1}{4}x - \frac{3}{4}y - z = \frac{1}{2}
\]

\[
\begin{align*}
\frac{1}{4}x + \frac{-3}{4}y + -1z &= \frac{1}{2}
\end{align*}
\]
10. (10 points) You don’t notice the hole in the dog food bag, and dog food is spilled across the kitchen floor, with density at point \((x, y)\) described by the (differentiable) dog food density function \(z = F(x, y)\). Your dog (Fido) comes flying into the kitchen and starts running in circles (i.e. along the path \(x(t) = \cos(t), y(t) = \sin(t)\)), eating the food.

(a.) (3 points) Give the chain rule formula to calculate \(dz/dt\).
\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

(b.) (7 points) Find \(dz/dt\) at \(t = \frac{\pi}{2}\). Data:
\[
\begin{align*}
F_x(1, 0) &= 5 & F_y(1, 0) &= 9 \\
F_x(0, 1) &= 6 & F_y(0, 1) &= -10 \\
F_x(\pi/2, \pi/2) &= 7 & F_y(\pi/2, \pi/2) &= 11 \\
F_x(-1, 0) &= 8 & F(-1, 0) &= 12
\end{align*}
\]
\[
\begin{align*}
\chi(x_1) &= 0 \\
y(\pi/2) &= 1
\end{align*}
\]
\[
\begin{align*}
\frac{dx}{dt}(\pi/2) &= -\sin(\pi/2) = 0 \\
\frac{dy}{dt}(\pi/2) &= \cos(\pi/2) = 0
\end{align*}
\]
\[
\frac{dx}{dt}(\pi/2) = F_x(0, 1)(-1) + F_y(0, 1)(0) = 6(-1) + 0 = -6 < 0
\]
\[
\frac{dz}{dt}(\pi/2) = -6
\]

(c.) (BONUS POINT) Is Fido feeling happy or sad at \(t = \pi/2\)? Circle one.

Happy \hspace{1cm} \text{Sad} \hspace{1cm} \text{HE'S GETTING LESS DOG FOOD}
11. (10 points) Find the graph of the function and write the letter in the space provided.

\[
\begin{align*}
    z &= |y| - |x| & z &= \cos\left(\sqrt{x^2 + y^2}\right) & z &= |xy| \\
    z &= -x^2 - y^2 & z &= x^3 \cos(y)
\end{align*}
\]

\[\text{G} \quad \text{B} \quad \text{J} \]

\[\text{L} \quad \text{E} \quad \text{E} \]

A. B. C. D. E. F. G. H. I. J. K. L. M.