Induced maps and bases

October 25, 2006

Let V have basis $\{e_j\}_1^k$ and W basis $\{f_i\}_1^p$. A linear map

$$T : V \to W$$

is given by the matrix $A = (A^i_j)$ with respect to the bases $\{e_j\}$ and $\{f_i\}$:

$$T(e_j) = \sum_i A^i_j f_i$$

and likewise for the dual

$$T^*(f^i) = \sum_j A^i_j e^j$$

1 Tensor algebra

The induced maps can be described in terms of the corresponding bases:

$$T_* : \mathcal{T}^0_r(V) \to \mathcal{T}^0_r(W)$$

$$\begin{aligned} T_*(e_{j_1} \otimes ... \otimes e_{j_r}) &= \left(\sum_{i_1} A^{i_1}_{j_1} f_{i_1}\right) \otimes ... \otimes \left(\sum_{i_r} A^{i_r}_{j_r} f_{i_r}\right) \\ &= \sum_{i_1,\ldots,i_r} A^{i_1}_{j_1} ... A^{i_r}_{j_r} f_{i_1} \otimes ... \otimes f_{i_r} \end{aligned}$$

We use the shorthand

$$A^{i_1 \ldots i_r}_{j_1 \ldots j_r} = A^{i_1}_{j_1} ... A^{i_r}_{j_r}$$

so that this becomes

$$T_*(e_{j_1} \otimes ... \otimes e_{j_r}) = \sum_{i_1,\ldots,i_r} A^{i_1 \ldots i_r}_{j_1 \ldots j_r} f_{i_1} \otimes ... \otimes f_{i_r}$$

Likewise, the induced map

$$T^* : \mathcal{T}^s_0(W) \to \mathcal{T}^s_0(V)$$

is given with respect to the basis by

$$T^*(f^{i_1} \otimes ... \otimes f^{i_s}) = \sum_{j_1,\ldots,j_s} A^{i_1 \ldots i_s}_{j_1 \ldots j_s} e^{j_1} \otimes ... \otimes e^{j_s}$$

2 Exterior algebra

The induced map on the exterior algebra can also be described in terms of bases. Let $e_{j_1} \wedge ... \wedge e_{j_r} \in \Lambda_r(V)$ be a basis vector. Then

$$T_*(e_{j_1} \wedge ... \wedge e_{j_r}) = \sum_{i_1,...,i_r} A^{i_1...i_r}_{j_1...j_r} f_{i_1} \wedge ... \wedge f_{i_r}$$

We note that we are not summing over our preferred basis since we don't have $i_1 < ... < i_r$, indeed, we are summing over all $i_1, ..., i_r$. First, since any term with some $i_q = i_t$, $q \neq t$, must be zero ($f_i \wedge f_i = 0$), we can write this as

$$T_*(e_{j_1} \wedge ... \wedge e_{j_r}) = \sum_{i_1,...,i_r; i_q \neq i_t} A^{i_1...i_r}_{j_1...j_r} f_{i_1} \wedge ... \wedge f_{i_r}$$

Therefore, for each $i_1, ..., i_r$ there is a permutation $\sigma_{i_1,...,i_r} = \sigma \in S_r$ so that

$$i_{\sigma(1)} < ... < i_{\sigma(r)}$$

So we have

$$T_*(e_{j_1} \wedge ... \wedge e_{j_r})$$

$$= \sum_{i_1,...,i_r; i_q \neq i_t} (-1)^{sgn(\sigma_{i_1,...,i_r})} A^{i_{\sigma_{i_1,...,i_r}(1)} \cdots i_{\sigma_{i_1,...,i_r}(r)}}_{j_{\sigma_{i_1,...,i_r}(1)} \cdots j_{\sigma_{i_1,...,i_r}(r)}} f_{i_{\sigma_{i_1,...,i_r}(1)}} \wedge ... \wedge f_{i_{\sigma_{i_1,...,i_r}(r)}}$$

This can be rewritten as

$$T_*(e_{j_1} \wedge ... \wedge e_{j_r}) = \sum_{i_1<...<i_r} \sum_{\sigma \in S_r} (-1)^{sgn(\sigma)} A^{i_1...i_r}_{j_{\sigma(1)} \cdots j_{\sigma(r)}} f_{i_1} \wedge ... \wedge f_{i_r}$$

Similarly, we can write

$$T^*(f^{i_1} \wedge ... \wedge f^{i_r}) = \sum_{j_1<...<j_r} \sum_{\sigma \in S_r} (-1)^{sgn(\sigma)} A^{i_{\sigma(1)} \cdots i_{\sigma(r)}}_{j_1...j_r} e^{j_1} \wedge ... \wedge e^{j_r}$$

If we let $I = i_1 < ... < i_r$ and $J = j_1 < ... < j_r$ be **multi-indices**, so that

$$e_J = e_{j_1} \wedge ... \wedge e_{j_r} \quad \text{and} \quad f^I = f^{i_1} \wedge ... \wedge f^{i_r}$$

and likewise for f_I and e^J. Then our expressions for T_* and T^* take on the simplified form

$$T_*(e_J) = \sum_I \sum_{\sigma \in S_r} (-1)^{sgn(\sigma)} A^I_{\sigma(J)} f_I$$

and

$$T^*(f^I) = \sum_J \sum_{\sigma \in S_r} (-1)^{sgn(\sigma)} A^{\sigma(I)}_J e^J$$

where $\sigma(J)$ and $\sigma(I)$ are the re-orderings of I and J, respectively, determined by σ.