Midterm 2, Math 417, Fall 2017

Name ___________________________  NetID ______________________

For full credit, show all your work and justify your answers unless otherwise instructed.
1. (8 points) Decide whether each statement below is true or false, and circle your answer. You do not need to justify your answer.

T  F  For complex numbers $z = re^{i\theta}$ and $w = se^{i\psi}$ we have $zw = rs(e^{i\theta} + e^{i\psi})$.

T  F  In any field $F$, the multiplicative inverse $a^{-1}$ of a nonzero element $a \in F$ is unique.

T  F  If $W \subset F[x]$ is any vector subspace of the space of polynomials with coefficients in a field $F$ and $f, g \in W$, then we necessarily have $fg \in W$.

T  F  The polynomial $f(x) = x^5 + \sqrt{3}x^4 + 2x^2 - \frac{3}{2}x \in \mathbb{R}[x]$ is irreducible.

T  F  The polynomial $g(x) = [1]x^3 + [1]x + [1] \in \mathbb{Z}_6[x]$ is irreducible.

T  F  The subset $X = \{ a + bi \mid a, b \in \mathbb{Z} \} \subset \mathbb{C}$ is a subring.

T  F  The symmetric group $S_3$ is isomorphic to the group $\mathbb{Z}_6$.

T  F  Every subgroup of a cyclic group is cyclic.
2. Suppose $\mathbb{F} \subset \mathbb{K}$ is a subfield and $y \in \mathbb{K}$ is a fixed element, and let

$$T : \mathbb{K} \rightarrow \mathbb{K}$$

be multiplication by $y$; that is, $T(x) = yx$.

(a) (2 points) Considering $\mathbb{K}$ as a vector space over $\mathbb{F}$, show that $T$ is a linear transformation (this should be a simple computation using the field axioms).

$$T(x_1 + x_2) = y(x_1 + x_2) = yx_1 + yx_2 = T(x_1) + T(x_2)$$

$$T(ax) = y(ax) = (y)(x) = yx = ax = a T(x)$$

(b) (2 points) In the special case of the subfield $Q \subset Q(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in Q\}$ and $y = 5 - 3\sqrt{2}$, write down the matrix for $T(x) = (5 - 3\sqrt{2})x$ in terms of the basis $1, \sqrt{2}$ for $Q(\sqrt{2})$ over $Q$.

$$\begin{pmatrix}
5 - 3\sqrt{2}
\end{pmatrix}
= 5 - 3\sqrt{2}$$

$$\begin{pmatrix}
5 - 3\sqrt{2} \sqrt{2}
\end{pmatrix}
= -6 + 5\sqrt{2}$$

$$\begin{pmatrix}
5 & -6 \\
-3 & 5
\end{pmatrix}$$
3. (2 points) Suppose \( f = x^3 + 2x^2 + x - 1, g = x^2 + 3x + 3 \in \mathbb{Q}[x] \). Find gcd\( (f, g) \) and express it in the form gcd\( (f, g) = uf + vg \) for \( u, v \in \mathbb{Q}[x] \).

\[
\begin{align*}
\text{gcd}(f, g) &= \frac{x-1}{x^2 + 3x + 3} \frac{x-1}{x^3 + 2x^2 + x - 1} \\
&= \frac{-x^2 - 2x - 1}{(x^3 + 3x^2 + 3x)} \\
&= \frac{x - 1}{x + 2}
\end{align*}
\]

\[
\begin{align*}
f - (x-1)g &= x + 2 \\
g - (x+2)(x+1) &= 1 \\
g - (f - (x-1)g)(x+1) &= 1 \\
g - (x+1)f + (x^2 - 1)g &= 1 \\
-x^2 + 1 &= 1
\end{align*}
\]
4. (3 points) A group is a nonempty set \( G \) with an operation \( * \) such that...
(List the three axioms for a group.)

I. \( * \) is associative

II. \( \exists e \in G \) such that \( \forall g \in G, \ e \ast g = g \ast e = g \)

III. \( \forall g \in G, \ \exists g^{-1} \in G \) such that \( g \ast g^{-1} = g^{-1} \ast g = e \)

5. Let \( G \) and \( H \) be groups.

(a) (1 point) In one sentence say what it means for \( \phi : G \to H \) to be a group homomorphism.

\[ \forall g, g' \in G, \text{ we have} \]
\[ \phi(gg') = \phi(g) \phi(g') \]

(b) (2 points) Prove that if \( \phi : G \to H \) is a homomorphism and if \( \ker(\phi) = \{e\} \), then \( \phi \) is injective (here \( e \) is the identity in \( G \)).

Suppose \( \phi(g) = \phi(g') \). Then \( e = \phi(g)^{-1} \phi(g') \)
\[ = \phi(g^{-1}) \phi(g') \]
\[ = \phi(g^{-1}g') \]

So, \( g^{-1}g' \in \ker(\phi) = \{e\} \), hence
\[ g^{-1}g' = e \]
\[ \text{or} \]
\[ g' = g \]
6. Consider the symmetric group $S_6$ consisting of permutations of the set \{1, 2, 3, 4, 5, 6\}, and let

\[ H = \{ \sigma \in S_6 \mid \sigma(3) = 3 \}. \]

(a) (2 points) Prove that $H$ is a subgroup of $S_6$.

\[ e = \text{id} \in S_6 \text{ has } \text{id}(3) = 3, \text{ so } H = e, \text{ and } H \neq \emptyset. \]

\[ \forall \sigma, \tau \in H, \text{ we have} \]

\[ \sigma \tau(3) = \sigma(\tau(3)) = \sigma(3) = 3, \text{ so } \sigma \tau \in H. \]

\[ \sigma^{-1}(3) = \sigma^{-1}(\sigma(3)) = \sigma^{-1}(3) = \text{id}(3) = 3. \]

(b) (1 point) Is $H$ a normal subgroup of $S_6$? Briefly justify your answer.

\[ \text{No;} \quad (12) \in H \text{ but} \]

\[ (13)(12)(13)^{-1} = (13)(12)(13) = (23) \notin H \]

(c) (1 point) Is $H$ isomorphic to $S_5$? Circle your answer. No justification required. \[ \text{YES} \quad \text{NO} \]

7. (2 point) Draw the subgroup lattice of $\mathbb{Z}_{20}$.

\[ \text{conjugation by } (36) \in S_6 \text{ gives an isomorphism to} \]

\[ \text{the subgroup } S_5 < S_6. \]