Let $K$ be a field and $p(x) \in K[x]$ an irreducible polynomial. We have shown in class that the quotient ring $F = K[x]/(p(x))$ is actually a field. The elements of $F$, which are images of elements of $K[x]$ by the quotient homomorphism $\pi: K[x] \to F$, are cosets of the additive group, and so have the form

$$\pi(f(x)) = f(x) + (p(x)).$$

We also showed (using long division) that we need only consider polynomials $f(x)$ with $\text{deg}(f) < \text{deg}(p)$. So, suppose $\text{deg}(p) = n$, then we have

$$F = \{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + (p(x)) \mid a_0, \ldots, a_{n-1} \in K\}.$$

Alternatively, setting $\alpha = \pi(x)$, we have

$$a_0 + a_1 x + \cdots a_{n-1} x^{n-1} + (p(x)) = \pi(a_0) + \pi(a_1) \alpha + \cdots \pi(a_{n-1}) \alpha^{n-1}.$$ 

As we pointed out in class, the restriction of $\pi$ to $K \subset K[x]$ is injective. Consequently, if we use this to view $K$ as a subfield of $F$ (meaning that we identify $a \in K$ with its image $\pi(a)$ in $F$), then the elements of $F$ take the simple form

$$a_0 + a_1 \alpha + \cdots a_{n-1} \alpha^{n-1}.$$ 

Thus, we can write

$$F = \{a_0 + a_1 \alpha + \cdots + a_{n-1} \alpha^{n-1} \mid a_0, \ldots, a_{n-1} \in K\}.$$ 

We have not shown that representing the elements of $F$ as $K$-linear combinations $a_0 + a_1 \alpha + \cdots + a_{n-1} \alpha^{n-1}$ is unique, but it is.... In fact, this is close to what you end up proving in HW 12, problem 4 (for that particular example).

The homomorphism $\pi$ also looks simpler in this way:

$$\pi(a_0 + a_1 x + \cdots a_m x^m) = a_0 + a_1 \alpha + \cdots + a_m \alpha^m.$$ 

Note however, that $m$ might be bigger than $n-1$, and so to write it as a sum of powers of $n-1$, you should first find a different coset representative (using long division again) whose degree is less than $n$.

Since we have identified $K \subset F$, every polynomial $g(x) = a_0 + a_1 x + \cdots a_m x^m \in K[x]$ can be viewed as a polynomial in $F[x]$ (since each $a_i \in K \subset F$). Doing this for the polynomial $p$ above, we showed that $\alpha$ is a root of $p$. To see this, write $p(x) = p_0 + p_1 x + \cdots + p_n x^n$, and then note that since $\pi$ is a homomorphism, we have

$$p(\alpha) = p_0 + p_1 \alpha + \cdots + p_n \alpha^n = \pi(p_0) + \pi(p_1) \pi(x) + \cdots \pi(p_n) p(x^n) = \pi(p(x)) = 0.$$