Section5.3 Exercise1:

We have to solve the equation $A^TA\mathbf{x} = A^T\mathbf{b}$.

a) In this case $A = \begin{pmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

Thus $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

b)In this case $A = \begin{pmatrix} -1 & 1 \\ 2 & 1 \\ 1 & -2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 10 \\ 5 \\ 20 \end{pmatrix}$.

Thus $\mathbf{x} = \begin{pmatrix} 19/7 \\ -26/7 \end{pmatrix}$.

c) In this case $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 2 \end{pmatrix}$.

Thus $\mathbf{x} = \begin{pmatrix} 8/5 \\ 3/5 \\ 6/5 \end{pmatrix}$. □

Section5.3 Exercise11:

a) We know $P = A(A^TA)^{-1}A^T$, thus

$$P^2 = (A(A^TA)^{-1}A^T)\cdot(A(A^TA)^{-1}A^T) = A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T = P.$$

b) We prove $P^n = P$ by induction. The $n = 1, 2$ cases are proven. For $n \geq 3$, we know $P^n = P^{n-1} \cdot P \overset{\text{induction}}{=} P \cdot P = P^2 \overset{\text{by a)}}{=} P$.

c) $P^T = (A(A^TA)^{-1}A^T)^T = (A^T)^T((A^TA)^{-1})^TA^T = A((A^TA)^T)^{-1}A^T = A(A^TA)^{-1}A^T = P$. Thus P is symmetric. □

Section5.3 Exercise12:

Expand the equation

$$\begin{pmatrix} A & I \\ 0 & A^T \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{x}} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix},$$

we get

$$A\widehat{\mathbf{x}} + \mathbf{r} = \mathbf{b}$$
$$A^T\mathbf{r} = 0.$$

Multiply A^T to both sides of first equation, we get $A^TA\widehat{\mathbf{x}} + A^T\mathbf{r} = A^T\mathbf{b}$. By second equation, we get $A^TA\widehat{\mathbf{x}} = A^T\mathbf{b}$. This means $\widehat{\mathbf{x}}$ is a least squares solution. Also, the first equation gives $\mathbf{r} = \mathbf{b} - A\widehat{\mathbf{x}}$, this means $\mathbf{r}$ is the residue for $\widehat{\mathbf{x}}$. □

Section5.4 Exercise4:

a) $\langle A, B\rangle = 1(-4) + 2(1) + 2(1) + 1(-3) + 0(3) + 2(2) + 3(1) + 1(-2) + 1(-2) = -4+2+2-3+4+3-2-2 = 0$.
b) Same computation, we have $\|A\|_F^2 = 1^2 + 2^2 + 2^2 + 1^2 + 2^2 + 3^2 + 1^2 + 1^2 = 1+4+4+1+4+9+1+1 = 25$. Thus $\|A\|_F = 5$.
c) $\|B\|_F = 7$.
d) $\|A+B\|_F = \sqrt{74}$. □

Section5.4 Exercise9:
First, we prove that $\sin nx$ and $\cos mx$ are orthogonal. We have $\sin n(-x)\cos m(-x) = -\sin nx \cos mx$, this means that $\sin nx \cos mx$ is an odd function. As a result, we have $\langle \sin nx, \cos mx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0$.

We now compute $\|\sin nx\|$. $\langle \sin nx, \sin nx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos 2nx}{2} dx$. Now since $\cos 2nx$ is an odd function, we don't need to integrate this part. Thus $\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos 2nx}{2} dx = \frac{1}{\pi}\int_{-\pi}^{\pi} 1/2 dx = 1$ and $\sin nx$ is a unit vector.

Also, $\langle \cos mx, \cos mx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos mx \cos mx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{(1+\cos 2mx)}{2} dx = 1$, since $\cos 2mx$ is odd.

As example 2 in page 248 shows, since $\sin nx$ and $\cos mx$ are orthogonal unit vectors, we have the distance between $\sin nx$ and $\cos mx$ is $\sqrt{2}$. □

Section5.4 Exercise29:
a) We have $\langle A\mathbf{x}, \mathbf{y}\rangle = (A\mathbf{x})^T\mathbf{y} = (\mathbf{x}^T A^T)\mathbf{y} = \mathbf{x}^T(A^T\mathbf{y}) = \langle \mathbf{x}, A^T\mathbf{y}\rangle$.
b) We have $\langle A^T A\mathbf{x}, x\rangle \overset{\text{by a)}}{=} \langle A\mathbf{x}, (A^T)^T\mathbf{x}\rangle = \langle A\mathbf{x}, A\mathbf{x}\rangle = \|A\mathbf{x}\|^2$. □