

**Section 4.3 Exercise 7:**

Since A is similar to B, there exists a nonsingular matrix S such that  $A = S^{-1}BS$ . Also since B is similar to C, there is a nonsingular matrix T such that  $B = T^{-1}CT$ . Then we know  $TS$  is nonsingular and  $A = S^{-1}(T^{-1}CT)S = (TS)^{-1}C(TS)$ . Thus A is similar to C.  $\square$

**Section 4.3 Exercise 10:**

Since A is similar to B, there exists a nonsingular matrix S such that  $A = S^{-1}BS$ . Now let  $T = BS^{-1}$ , we then have  $A = ST$  and  $B = TS$  as required.  $\square$

**Section 5.1 Exercise 11:**

a) We have  $\mathbf{x}^T \mathbf{x} = (x_1^2 + x_2^2) \geq 0$ .

b) We have  $\mathbf{x}^T \mathbf{y} = x_1 y_1 + x_2 y_2 = y_1 x_1 + y_2 x_2 = \mathbf{y}^T \mathbf{x}$ .

c) We have  $\mathbf{x}^T (\mathbf{y} + \mathbf{z}) = \mathbf{x}^T (y_1 + z_1, y_2 + z_2) = x_1(y_1 + z_1) + x_2(y_2 + z_2) = x_1 y_1 + x_2 y_2 + x_1 z_1 + x_2 z_2 = \mathbf{x}^T \mathbf{y} + \mathbf{x}^T \mathbf{z}$ .  $\square$

**Section 5.1 Exercise 14:**

a) We know  $\alpha^2 + h^2 = \|\mathbf{a}_1\|^2$  and  $\alpha = \frac{\mathbf{a}_1^T \mathbf{a}_2}{\|\mathbf{a}_2\|}$ . Using these two identities we get  $h^2 \|\mathbf{a}_2\|^2 = (\|\mathbf{a}_1\|^2 - (\frac{\mathbf{a}_1^T \mathbf{a}_2}{\|\mathbf{a}_2\|})^2) \|\mathbf{a}_2\|^2 = \|\mathbf{a}_1\|^2 \|\mathbf{a}_2\|^2 - (\mathbf{a}_1^T \mathbf{a}_2)^2$ .

b) Write  $\mathbf{a}_1 = (a_{11}, a_{12})$  and  $\mathbf{a}_2 = (a_{21}, a_{22})$ . We get  $h^2 \|\mathbf{a}_2\|^2 = (a_{11}^2 + a_{12}^2)(a_{21}^2 + a_{22}^2) - (a_{11}a_{21} + a_{12}a_{22})^2 = (a_{11}a_{22} - a_{21}a_{12})^2 = |\det(A)|^2$ . Hence Area P =  $h\|\mathbf{a}_2\| = |\det(A)|$ .  $\square$

**Section 5.2 Exercise 6:**

It is impossible. If such matrix A exists, then we know  $(3, 1, 2)^T$  is in column space of  $A^T$ , which is  $R(A^T)$  and we have  $(2, 1, 1)^T \in N(A)$ . However these two subspaces are orthogonal to each other, but

$$\langle (3, 1, 2)^T, (2, 1, 1)^T \rangle = 9 \neq 0.$$

$\square$

**Section 5.2 Exercise 16:**

First we recall from page 232 that  $R(A) = \{A\mathbf{y} | \mathbf{y} \in R(A^T)\}$ , since  $\{\mathbf{x}_1, \dots, \mathbf{x}_r\}$  spans  $R(A^T)$  we know  $\{A\mathbf{x}_1, \dots, A\mathbf{x}_r\}$  spans  $R(A)$ . Now we have to prove that  $\{A\mathbf{x}_1, \dots, A\mathbf{x}_r\}$  is linear independent. Let  $0 = \sum_{i=1}^r c_i A\mathbf{x}_i = A(\sum c_i \mathbf{x}_i)$  this shows  $\sum c_i \mathbf{x}_i \in N(A)$ . Also we know  $\sum c_i \mathbf{x}_i \in R(A^T)$  because each  $\mathbf{x}_i$  is in  $R(A^T)$ . Thus we know  $\sum c_i \mathbf{x}_i \in N(A) \cap R(A^T) = \{0\}$ , that is  $\sum c_i \mathbf{x}_i = 0$ . Which implies  $c_i = 0$  for all  $i$  since  $\{\mathbf{x}_i\}$  is linear independent. This shows that  $\{A\mathbf{x}_1, \dots, A\mathbf{x}_r\}$  is linear independent.  $\square$