Section4.2 Exercise3:

Let $\mathbf{e_1}, \mathbf{e_2}$ and $\mathbf{e_3}$ be the standard basis for $\mathbb{R}^3$.

a)Given $L(x_1, x_2, x_3)^T = (x_3, x_2, x_1)^T$. We have $L(\mathbf{e_1}) = \mathbf{e_3}$, $L(\mathbf{e_2}) = \mathbf{e_2}$, and $L(\mathbf{e_3}) = \mathbf{e_1}$. Thus

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

b) For $L(x_1, x_2, x_3)^T = (x_1, x_1 + x_2, x_1 + x_2 + x_3)^T$, we have $L(\mathbf{e_1}) = (1, 1, 1)^T, L(\mathbf{e_2}) = (0, 1, 1)^T, L(\mathbf{e_3}) = (0, 0, 1)^T$. Thus

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

c) This time $L(x_1, x_2, x_3) = (2x_3, x_2 + 3x_1, 2x_1 - x_3)$. Thus $L(\mathbf{e_1}) = (0, 3, 2)^T, L(\mathbf{e_2}) = (0, 1, 0), L(\mathbf{e_3}) = (2, 0, -1)^T$. And

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$

□

Section4.2 Exercise13:

We know $\{1, x\}$ is a basis for P_2. Thus the requested matrix A is indeed the matrix representation of L with respect to this basis $\{1, x\}$ and the standard basis of $\mathbb{R}^2$. We have

$$L(1) = \begin{pmatrix} \int_0^1 (1)dx \\ 1(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and

$$L(x) = \begin{pmatrix} \int_0^1 (x)dx \\ x(0) \end{pmatrix} = \begin{pmatrix} 1/2(1-0) \\ 0 \end{pmatrix},$$

where $1(0), x(0)$ is the evaluation at 0 of the two polynomials 1 and x. Thus

$$A = \begin{pmatrix} 1 & 1/2 \\ 1 & 0 \end{pmatrix}.$$

□

Section4.2 Exercise14:

Let $b_1 = 2 = (2, 0)^T, b_2 = 1 - x = (1, -1)^T$. And let $L(p(x)) = p'(x) + p(0)$, then we have

$$L(\vec{u_1}) = L(x^2) = 0 \cdot 1 + 2 \cdot x,\ L(\vec{u_2}) = L(x) = 1 + 0 \cdot x,\ L(\vec{u_3}) = L(1) = 1 + 0 \cdot x.$$

Thus we have matrix

$$B = \begin{pmatrix} \vec{b_1} & \vec{b_2} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}, L = \begin{pmatrix} \vec{u_1} & \vec{u_2} & \vec{u_3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$

Thus $A = B^{-1}L = -1/2 \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix} L = \begin{pmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{pmatrix}$.
a) $L(x^2 + 2x - 3) = L(1, 2, -3)^T = A(1, 2, -3)^T = (1/2, -2)^T$.
b) $L(x^2 + 1) = A(1, 0, 1)^T = (3/2, -2)^T$.
c) $L(3x) = A(0, 3, 0)^T = (3/2, 0)^T$.
d) $L(4x^2 + 2x) = A(4, 2, 0) = (5, -8)^T$. □

Section4.2 Exercise16:
Since A is the standard matrix representation of L. For the given vector $0 \neq \mathbf{x} \in \mathbb{R}^n$ we have by theorem 4.2.1 that $0 = L(\mathbf{x}) = A\mathbf{x}$, thus A is singular by theorem 1.4.2. □

Section4.2 Exercise20:
For this question we use the Matrix Representation theorem 4.2.2.
a) We have $[L(v)]_F = A[v]_E$, thus $v \in ker(L) \iff 0 = [L(v)]_F = A[v]_E \iff [v]_E \in N(A)$. □

Remark(see textbook page 157): The coordinate vector of the zero vector $\mathbf{0}$ of some vector space is always the zero vector in $\mathbb{R}^n$ and vice versa. Thus we can write $v \in ker(L) \iff 0 = L(v) \iff 0 = [L(v)]_F$. Here the first 0 is the zero vector in W and the second one is in $\mathbb{R}^{dim\,W}$. This remark makes above paragraph more complete but not necessary to write down, I think.

b) We know $w \in L(V)$ if and only if that $\exists v \in V$ such that $w = L(v)$ or equivalently, $[w]_F = [L(v)]_F = A[v]_E$. And the last expression is equivalent to $[w]_F$ is in the column space of A. □