**Section3.5 Exercise5:**

**a)** The transition matrix from $[e_1, e_2, e_3]$ to $[u_1, u_2, u_3]$ is

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

**b)** The coordinates wrt $[u_1, u_2, u_3]$ are:

**i)** $A(3,2,5)^T = (1,-4,3)^T$; **ii)** $A(1,1,2)^T = (0,-1,1)^T$; **iii)** $A(3,3,2)^T = (2,2,-1)^T$.

□

**Section3.6 Exercise10:**

Recall that dimension of row space equals to the dimension of column space and that row equivalent matrices have the same rank, we have rank of A = rank of U = 2. Also notice that $\mathbf{a_1}$ and $\mathbf{a_2}$ are linear independent, hence $\mathbf{a_3}$ and $\mathbf{a_4}$ are linear combinations of $\mathbf{a_1}$ and $\mathbf{a_2}$. Thus we can write $\mathbf{a_3} = x\mathbf{a_1} + y\mathbf{a_2}, \mathbf{a_4} = z\mathbf{a_1} + w\mathbf{a_2}$ for some $x, y, z, w \in \mathbb{R}$:

$$A = \begin{pmatrix} -3 & 4 & -3x+4y & -3z+4w \\ 5 & -3 & 5x-3y & 5z-3w \\ 2 & 7 & 2x+7y & 2z+7w \\ 1 & -1 & x-y & z-w \end{pmatrix}$$

Now since A and U are row equivalent, they have the same row space. Thus $(-3, 4, -3x+4y, -3z+4w), (5, -3, 5x-3y, 5z-3w)$ are linear combinations of $(1, 0, 2, 1)$ and $(0, 1, 1, 4)$. We have $(-3, 4, -3x+4y, -3z+4w) = a(1, 0, 2, 1) + b(0, 1, 1, 4)$ and $(5, -3, 5x-3y, 5z-3w) = c(1, 0, 2, 1) + d(0, 1, 1, 4)$ for some $a, b, c, d \in \mathbb{R}$. Comparing the first two coordinates in these two equations, we see immediately that $a = -3, b = 4, c = 5, d = -3$. Substituting them back and looking the rest two coordinates, we have

$$\begin{aligned} -3x + 4y &= 2 \cdot (-3) + 1 \cdot (4 \\ -3z + 4w &= 1 \cdot (-3) + 4 \cdot (4) \\ 5x - 3y &= 2 \cdot (5) + 1 \cdot (-3) \\ 5z - 3w &= 1 \cdot (5) + 4 \cdot (-3). \end{aligned}$$

Solving these equations, we have $x = 2, y = 1, z = 1, w = 4$. Therefore $\mathbf{a_3} = (-2, 7, 11, 1)^T$ and $\mathbf{a_4} = (13, -7, 30, -3)^T$. □

**Section3.6 Exercise22:**

**a)**To show the column space of $C = AB$ is a subspace of the column space of $A$, it suffices to show that each column vector of $C$ is a linear combination of column vectors of $A$. Now let $\mathbf{b_i}$ be the $i-th$ column vector of $B$. Then we know the $i-th$ column vector of $C = AB$ is $A\mathbf{b_i}$. (See textbook page 38). Now we write $A = (\mathbf{a_1}, ..., \mathbf{a_n})$ where

each $\mathbf{a_i}$ is a column vector. And write down $\mathbf{b_i} = (b_{i1}, ..., b_{in})^T$ explicitly. In this way, we have $A\mathbf{b_i} = b_{i1}\mathbf{a_1} + ... + b_{in}\mathbf{a_n}$ which is a linear combination of column vectors of $A$. Thus we proved **a**.
**b)** We can prove this in two ways. First we can prove it similar to **a)**. Or we can apply **a)** to $C^T = B^T A^T$. The fact that column space of one matrix is the row space of its transpose gives us a second proof.
**c)** By **a)** Column rank of $C \leq$ Column rank of $A$. By **b)** Row rank of $C \leq$ Row rank of $B$. In fact we know that rank= column rank =row rank. Thus $rank(C) \leq min\{rank(A), rank(B)\}$. □

**Section4.1 Exercise3:**
We have $L(\mathbf{0}) = \mathbf{a}$, which is a given nonzero vector. Hence $L$ can not be a linear transformation. □

**Section4.1 Exercise8:**
**a) Yes:** $L(aA + B) = C(aA + B) + (aA + B)C = aCA + CB + aAC + BC = a(CA + AC) + (CB + BC) = aL(A) + L(B)$.
**b) Yes:** Similar computation.
**c) No, in general:** If $L$ were a linear operator, then we have $0 = L(I + (-I)) = L(I) + L(-I) = I^2C + (-I)^2C = 2C$, which isn't always true. If fact, $L$ is a linear operator if and only if $C = 0$, the zero matrix. □

**Section4.1 Exercise16:**
Let's verify:

$$\begin{aligned} L(a\mathbf{u_1} + \mathbf{u_2}) &= \mathbf{L_2}(\mathbf{L_1}(\mathbf{au_1} + \mathbf{u_2})) && \text{because} L = L_2(L_1) \\ &= L_2(aL_1(\mathbf{u_1}) + \mathbf{L_1}(\mathbf{u_2})) && \text{because } L_1 \text{ is a linear operator} \\ &= aL_2(L_1(\mathbf{u_1})) + \mathbf{L_2}(\mathbf{L_1}(\mathbf{u_2})) && \text{because } L_2 \text{ is a linear operator} \\ &= aL(\mathbf{u_1}) + \mathbf{L}(\mathbf{u_2}) && L = L_2(L_1). \end{aligned}$$

Thus $L$ is a linear operator. □