Section6.4 Exercise13:
Given $U = I - 2uu^H$, by exercise 7 we have $U^H = (I - 2uu^H)^H = I^H - \bar{2}u^{H^H}u^H = I - 2uu^H = U$. This shows U is Hermitian.

$U^H U = (I-2uu^H)(I-2uu^H) = I-2uu^H-2uu^H+4(uu^H)(uu^H)$. Now u is a unit vector, in other words $u^H u = 1$, hence $U^H U = I - 4uu^H + 4u(u^H u)u^H = I - 4uu^H + 4uu^H = I$. This shows U is unitary.

Finally, from $U^H U = I$ we know $U^{-1} = U^H = U$. That is U is its own inverse. □

Section6.5 Exercise1:
Let $A = U\Sigma V^T$ be a singular value decomposition. Observe that Σ^T is a matrix whose off-diagonal entries are all 0's and whose diagonal entries are exactly the diagonal entries of Σ. In particular, diagonal entries are decreasing and positive. Thus $A^T = V\Sigma^T U^T$ is a singular value decomposition of A^T. Hence by the uniqueness of singular values, we know the singular values of A^T are the diagonal entries of $\Sigma^T = \Sigma$, which are also singular values of A. □

Section6.5 Exercise2:
a) Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, then $A^T A = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$ which has eigenvalues $\lambda_1 = 10, \lambda_2 = 0$. Hence we get the singular values of A $\sigma_1 = \sqrt{10}, \sigma_2 = 0$.(remember: we must pick $\lambda_1 \geq \lambda_2$)
We now compute the matrix V. For this we first need to find out the eigenspaces of $A^T A$. For $\lambda_1 = 10$, the eigenspace is clearly $V_1 = span\{(1,1)^T\}$. For $\lambda_2 = 0$, the eigenspace is $V_2 = span\{(1,-1)^T\}$. Normalize these two eigenvectors, we get $V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.
Turning onto get U. First from observation 4, we have

$$\mathbf{u_1} = A\mathbf{v_1}/\sigma_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}.$$

Now since the second singular value $\sigma_2 = 0$, observation 4 tell us no info about $\mathbf{u_2}$, just like the example 1 case. We need other observations. Since A has rank 1, by observation 5 $\mathbf{u_2}$ is an orthonormal basis for $N(A^T) = \{(x,y)^T | x + 2y = 0\} = span\{(2,-1)^T\}$. Normalizing this vector, we have

$$\mathbf{u_2} = \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}; \quad \texttt{and set} \quad U = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix}.$$

The singular value decomposition is

$$A = U\begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{pmatrix}V^T.\square$$

c) Let

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \texttt{then we have} \quad A^T A = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$$

Eigenvalues for A^TA are $\lambda_1 = 16, \lambda_2 = 4$, and singular values of A are $\sigma_1 = 4, \sigma_2 = 2$. For $\lambda_1 = 16$, the eigenspace is $V_1 = \{(x,y)^T | -6x+6y=0\} = span\{(1,1)^T\}$. And for $\lambda_2 = 4$, the eigenspace is $V_2 = \{(x,y)^T | 6x+6y=0\} = span\{(1,-1)^T\}$. Normalizing these two eigenvectors, we set

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since both the singular values are all nonzero, we can use full strength of observation 4:

$$\mathbf{u_1} = \frac{1}{4} A \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \texttt{and} \quad \mathbf{u_2} = \frac{1}{2} A \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Again since A has rank 2, we can use observation 5 to get the rest two column vectors of U. The null space $N(A^T) = \{(x,y,z,w)^T | x+3y = 0 = 3x+y\} = \{(0,0,z,w)^T\}$, which has a natural orthonormal basis: $\mathbf{u_3} = (0,0,1,0)^T, \mathbf{u_4} = (0,0,0,1)^T$. Thus we can set

$$U = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The singular value decomposition is

$$A = U \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} V^T. \Box$$

Section6.5 Exercise8:

Let $A = U\Sigma V^T$ be a singular value decomposition. Notice that since A is a square $n \times n$ matirx, U, Σ and V are all $n \times n$ matrices. Moreover, since Σ is a diagonal square matrix we have $\Sigma^T = \Sigma$. Computations give $A^T = V\Sigma^T U^T = V\Sigma U^T$, $AA^T = (U\Sigma V^T)(V\Sigma U^T) = U\Sigma^2 U^T$ and $A^TA = V\Sigma^2 V^T$ since $UU^T = VV^T = I$. (because A is a square matrix, UU^T and VV^T give identity matirices of the rank) Define $X = VU^{-1} = VU^T$. Since both U and V are invertible, X is invertible and $X^{-1} = (VU^{-1})^{-1} = UV^{-1} = UV^T$. Finally $XAA^TX^{-1} = (VU^T)(U\Sigma^2U^T)UV^T = VUU^T\Sigma^2U^TUV^T = V\Sigma^2V^T = A^TA$, this shows A^TA and AA^T are similar. $\Box$

Section6.6 Exercise1:

$$\textbf{a)} \begin{pmatrix} 3 & -5/2 \\ -5/2 & 1 \end{pmatrix}; \ \textbf{b)} \begin{pmatrix} 2 & 1/2 & -1 \\ 1/2 & 3 & 3/2 \\ -1 & 3/2 & 1 \end{pmatrix}; \ \textbf{c)} \begin{pmatrix} 1 & 1/2 & -1 \\ 1/2 & 2 & 3/2 \\ -1 & 3/2 & 1 \end{pmatrix} \Box$$

Section6.6 Exercise4:
Since λ_1 and λ_2 are eigenvalues, a suitable change of coordinates makes the original equation $ax^2 + 2bxy + cy^2 = 1$ into this new equation $\lambda_1 x^2 + \lambda_2 y^2 = f$ for some constant f. We know $\lambda_1 \lambda_2 < 0$, which means they differ in sign. Thus this equation represents a hyperbola. $\square$

Section6.6 Exercise7:
a)Let $f(x, y) = 3x^2 - xy + y^2$ and $P = (0, 0)$. Then $f_x(x, y) = 6x - y$ and $f_y = -x + 2y$. Because $f_x(P) = f_y(P) = 0$ we know P is a stationary point. Continuing computations give $f_{xx} = 6, f_{xy} = f_{yx} = -1, f_{yy} = 2$. Thus the Hessian of f at P is $H = \begin{pmatrix} 6 & -1 \\ -1 & 2 \end{pmatrix}$, which has the characteristic polynomial $f(x) = x^2 - 8x + 11 = (x - (4 + \sqrt{5}))(x - (4 - \sqrt{5}))$. Since $(4 \pm \sqrt{5}) > 0$ we know H is positive definite and $f(P)$ is a local minimum.
e)Let $f(x, y, z) = x^3 + xyz + y^2 - 3x$ and $P = (1, 0, 0)$. We have $f_x = 3x^2 + yz - 3$, $f_y = xz + 2y$, $f_z = xy$ and $f_x(P) = f_y(P) = f_z(P) = 0$, hence P is a stationary point. Also we have $f_{xx} = 6x$, $f_{xy} = f_{yx} = z$, $f_{xz} = f_{zx} = y$, $f_{yy} = 2$, $f_{yz} = f_{zy} = x$, $f_{zz} = 0$, thus the Hessian of f at P is $H = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ which has the characteristic polynomial $(6 - x)(x^2 - 2x - 1)$ and eigenvalues $6, 1 \pm \sqrt{2}$. Since eigenvalues differ in sign, P is a saddle point. $\square$