
NAME SOLNS NetID

Circle your discussion section:

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<th>HD1: TR 9:00 – 9:50</th>
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<tbody>
<tr>
<td>Dane Skabelund</td>
<td></td>
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<td>Dileep Menon</td>
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• Closed book and notes.

• No phones, calculators, or electronic devices of any kind.

• For full credit, **show all your work and justify your answers for every problem. Do not erase your work!**

• You can use the last page for scratch work. This will not be graded.
1. Consider the transformation \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) given by \( T(u, v) = (uv^2, v - u) \).

(a) Circle the Jacobian of \( T \): (2 points)

\[
|dT| = \begin{vmatrix}
v^2 & 2uv \\
-1 & 1
\end{vmatrix} = v^2 + 2uv
\]

\[
|dT| = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix}
u^2 + v^2 & u^2 - v^2 \\
u^2 + 2uv & 2uv
\end{vmatrix}
\]

(b) Let \( S \) be the square in the \((u, v)\)-plane where \( 0 \leq u \leq 1 \) and \( 0 \leq v \leq 1 \). Find the area of its image \( T(S) \) in the \((x, y)\)-plane. (3 points)

\[
\text{Area}(T(S)) = \iint_{T(S)} dA = \iint_{S} (v^2 + 2uv) \, dudv
\]

\[
= \int_{0}^{1} \int_{0}^{1} (uv^2 + u^2v) \, dv \, du
\]

\[
= \int_{0}^{1} \left. \left( \frac{uv^2}{2} + \frac{u^2v^2}{2} \right) \right|_{0}^{1} \, dv
\]

\[
= \int_{0}^{1} \left( \frac{v^2}{2} + \frac{v^2}{2} \right) \, dv = \frac{1}{2} + \frac{1}{2} = \frac{5}{6}
\]

\[\text{Area}(T(S)) = \frac{5}{6}\]
2. Suppose the region $E$ where $x^2 + y^2 \leq z \leq 4$ and $y \geq 0$ is made of material whose density is given by $\rho(x, y, z) = z^2$.

(a) Fill in the limits and integrands of the two integrals below so that they each compute the mass of $E$, first in rectangular coordinates, then in cylindrical coordinates. (10 points)

\[
\int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_0^{\sqrt{z} - x^2} z^2 \, dy \, dx \, dz
\]

\[
\int_0^{\pi} \int_0^2 \int_0^{\sqrt{z}} z^2 \, dz \, dr \, d\theta
\]

(b) Circle the center of mass of $E$, whose coordinates have been rounded to one decimal place. Note: This can be done without evaluating any integrals. (1 point)

(0, 0.7, 1) (0, 0.7, 2) (0, 0.7, 3) (0.7, 0, 1) (0.7, 0, 2) (0.7, 0, 3)
3. Label the boxes next to the solid regions corresponding to the following two integrals: (2 points each)

(A) \[ \int_0^1 \int_z^1 \int_0^{y-z} f(x, y, z) \, dx \, dy \, dz \]

(B) \[ \int_0^1 \int_0^{1-z} \int_0^{1-y-z} g(x, y, z) \, dx \, dy \, dz \]

4. Consider the solid \( E = \{ x^2 + y^2 + z^2 \leq 4 \text{ and } y \geq 0 \text{ and } z \leq 0 \} \) in \( \mathbb{R}^3 \). Circle the correct limits of integration and check the box for the correct integrand that computes the given integral in spherical coordinates. (5 points)

\[
\iiint_E xy \, dV = \int_0^2 \int_0^{\pi} \int_0^{\frac{3\pi}{2}} \frac{3\pi}{2} 2\pi d\rho d\theta d\phi
\]

\[
\begin{array}{c}
\rho^3 \sin^3 \phi \cos \phi \\
\rho^3 \sin^3 \phi \cos \theta \sin \theta \\
\rho^4 \sin^2 \phi \sin \theta \\
\rho^4 \sin^3 \phi \sin \theta \cos \theta \\
\end{array}
\]

\( x y \rho^2 \sin \phi = \rho^4 \cos \phi \sin \phi \sin \phi \sin \phi \)
5. Consider the vector field \( \mathbf{F}(x, y) = \langle y \sin x - 2y, -\cos x \rangle \). Let \( R \) be the region in the plane of area 5 shown below, and let \( C \) be the boundary of \( R \), oriented as shown.

Compute \( \int_C \mathbf{F} \cdot d\mathbf{r} \). (4 points)

\[
\text{curl}(\mathbf{F}) = \sin x - (\sin x - 2) \mathbf{k} = 2 \mathbf{k}
\]

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_R \text{curl}(\mathbf{F}) \cdot \mathbf{k} \, dA = -2 \int_R dA = -2(5) = -10
\]

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = -10
\]

6. Consider the vector field \( \mathbf{F} = \langle xy^2z^3, e^{xyz}, z \rangle \).

(a) Compute the divergence and curl of the \( \text{(3 points)} \).

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
xy^2z^3 & e^{xyz} & -z
\end{vmatrix} = \langle -xye^{xyz}, 3xy^2z^2, ye^{xyz} - 2xyz \rangle
\]

\[
\text{div}(\mathbf{F}) = y^2z^3 + xze^{xyz} + 1
\]

\[
\text{curl}(\mathbf{F}) = \langle -xye^{xyz}, 3xy^2z^2, ye^{xyz} - 2xyz \rangle
\]

(b) \( \mathbf{F} \) is conservative \( \text{(1 point)} \).  

\[
\text{True} \quad \text{False} \quad \text{Not enough information}
\]

\[
\mathbf{F} = \nabla f \implies \text{curl}(\mathbf{F}) = \text{curl}(\nabla f) = \mathbf{0}
\]
7. Let $S$ be the surface parameterized by $\mathbf{r}(u, v) = \langle uv, u, v \rangle$ for $0 \leq u \leq 1$ and $0 \leq v \leq 1$.

(a) Mark the box next to the picture of $S$ below. (2 points)

(b) Fill in the integrand below to give a double integral in $u$ and $v$ that evaluates the surface integral $\iint_S xyz \, dS$. Do not evaluate the resulting integral. (3 points)

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v & 1 & 0 \\ u & 0 & 1 \end{vmatrix} = \langle 1, -v, -u \rangle$$

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{1 + v^2 + u^2}$$

$$\int_0^1 \int_0^1 u^2 v^2 \sqrt{1 + v^2 + u^2} \, dudv$$

(c) Using the orientation from the parameterization, fill in the integrand below to give a double integral in $u$ and $v$ that evaluates the flux $\iint_S (yz - x, z, y) \cdot dS$. Do not evaluate the resulting integral. (3 points)

$$\langle yz - x, z, y \rangle \cdot \mathbf{r}_u \times \mathbf{r}_v = \langle 0, v, u \rangle \cdot \langle 1, -v, -u \rangle = -v^2 - u^2$$

$$\int_0^1 \int_0^1 (-v^2 - u^2) \, dudv$$
8. (a) Let $M$ be the surface of revolution obtained by revolving the curve $z = y^2 + 1$, for $0 \leq y \leq 1$ about the $y$–axis shown below. Parameterize it by $r: D \rightarrow \mathbb{R}^3$, so that the grid lines are as shown, being sure to specify the domain $D$ of the parameterization in the $(u, v)$–plane. (3 points)

$$r(u, v) = \left( (u^2 + 1)^{1/2} \sin v, u, (u^2 + 1) \cos v \right)$$

$$D = \left\{ (u, v) \mid u \geq 0, 0 \leq v \leq 2\pi \right\}$$

(b) The surface integral $\iiint_M z \, dS$ is: negative zero positive (1 point)

(c) Orienting $M$ with normal $\mathbf{n}$ having positive second component, the flux

$$\iint_M \langle x, 0, z \rangle \cdot dS$$

is: negative zero positive (1 point)