NonBilipschitz Embeddability into RNP Spaces: Thick Families of Geodesics and Differentiation

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Overview of Talk

Background

- Banach spaces having the Radon-Nikodým property (RNP space).
- Differentiation based proofs of non-RNP biLipschitz embeddability of metric measure spaces.
- Thick families of geodesics and metric characterization of RNP.

New Results

- A "scale-specific" type of RNP differentiation on metric spaces containing a thick families of geodesics.
- An application to non-biLipschitz embeddability.
- Embedding spaces with true RNP Lipschitz differentiable structure into nonRNP spaces.

Background: RNP spaces

Definition (Radon-Nikodým property)

A Banach space V has the **Radon-Nikodým property** (RNP) if every Lipschitz map $\mathbb{R} \to V$ is differentiable Lebesgue-almost everywhere. In this case we call V an **RNP space**.

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- $\mathbb{R}$ - by Lebesgue's fundamental theorem. As a corollary - all finite dimensional normed spaces.
- Reflexive spaces, such as ℓ^p, L^p, $1 < p < \infty$.
- Separable dual spaces, such as $\ell^1 = c_0^*$.

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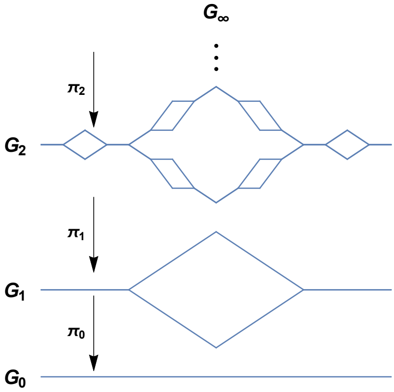
Nonexamples

- L^1, $t \mapsto \chi_{[0,t]}$, and c_0, $t \mapsto (\sin(nt)/n)_{n=1}^{\infty}$ are nowhere differentiable Lipschitz maps.

Background: Differentiation in Met. Meas. Spaces

Laakso-Lang-Plaut Infinite Diamond Graph, G_∞.



Theorem (Cheeger-Kleiner '09)

(1) For any RNP space V and Lipschitz map $f : G_\infty \to V$, f is differentiable wrt $\pi_0 : G_\infty \to G_0 \subseteq \mathbb{R}$ at μ_∞-a.e. $x \in G_\infty$, meaning there exists $f'(x) \in \mathbb{R}$ such that

$$f(y) - f(x) = f'(x)(\pi_0(y) - \pi_0(x)) + o(d(y,x)) \quad \text{as } y \to x$$

(2) Consequently, G_∞ does not biLipschitz embed into any RNP space.

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$$f(y) - f(x) = \quad 0 \qquad + o(d(y,x)) \quad \text{as } \pi_0^{-1}(\pi_0(x)) \ni y \to x$$

Which implies f is not biLipschitz! □

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- Uniform p-convexity (Mendel-Naor '13)

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Theorem (Ostrovskii '14a)

A Banach space V does not have the RNP if and only if it contains a biLipschitz copy of a thick family of geodesics.

Proof of $\Leftarrow$ is very natural, and does not use differentiation theory of Cheeger-Kleiner '09. Ostrovskii directly constructs an L^∞-bounded, L^1-divergent martingale, which is equivalent to nonRNP.

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Let (X, d) be a metric space, $u, v \in X$, and Γ a family of geodesics connecting u to v. Γ is **thick** if
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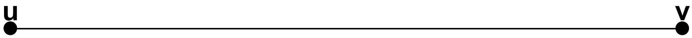

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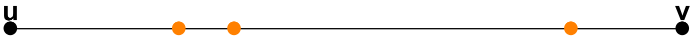

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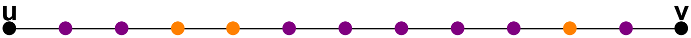

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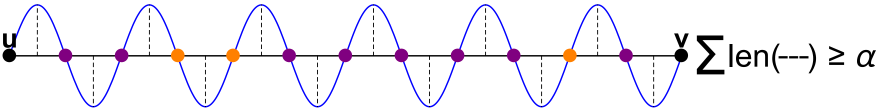
there is a superset of points $0 = t'_0 < t'_1 < \dots t'_{k'} = d(u, v)$
and another geodesic $\tilde{\gamma} \in \Gamma$ with
$\gamma(t'_i) = \tilde{\gamma}(t'_i)$ and
$\sum_{i=1}^{k'} \max_{t \in [t'_{i-1}, t'_i]} d(\gamma(t), \tilde{\gamma}(t)) \geq \alpha$.



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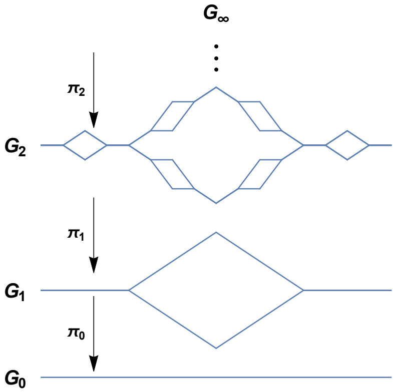
Example 1:



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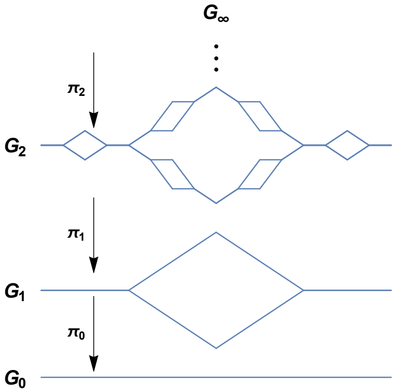
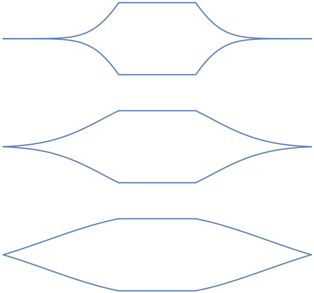


Example 2:

Non-quasiconvex deformation of diamond graphs. Replace edges with increasingly cuspidated diamonds.



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- Natural followup question to Ostrovskii's theorem: Does the biLipschitz containment of a thick family of geodesics characterize geodesic metric spaces nonembeddable into RNP spaces, like it does for Banach spaces?

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- $\mathbb{H}$ equipped with Carnot-Caratheodory metric does not biLipschitz embed into any RNP space (Cheeger-Kleiner '06, Lee-Naor '05, Semmes '96, Pansu '89).

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- $\mathbb{H}$ equipped with Carnot-Caratheodory metric does not biLipschitz embed into any RNP space (Cheeger-Kleiner '06, Lee-Naor '05, Semmes '96, Pansu '89).
- For $p > 0$, call a metric space (X, d) ***p*-convex** if there is a quasimetric ρ equivalent to d satisfying $\rho(w,y)^p/2 + \rho(z,y)^p/2 + \rho(x,y)^p - (\rho(w,x)/2)^p - (\rho(z,x)/2)^p \gtrsim \rho(w,z)^p \qquad \text{for all } w,x,y,z \in X.$
- $\mathbb{H}$ is 8-convex (Li '14). Later, $\mathbb{H}$ is 4-convex (Li '16).
- p-convex metric spaces do not contain biLipschitz copies of thick families of geodesics (Ostrovskii '14b).

Motivating Question

- The proof that $\mathbb{H}$ and the original proof that G_∞ do not embed into any RNP space uses a differentiation method.

- $\mathbb{H}$ does not contain a biLipschitz copy of thick family of geodesics, but do thick families of geodesics satisfy a differentiation theorem?

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- The proof that $\mathbb{H}$ and the original proof that G_∞ do not embed into any RNP space uses a differentiation method.
- $\mathbb{H}$ does not contain a biLipschitz copy of thick family of geodesics, but do thick families of geodesics satisfy a differentiation theorem?
- Yes, but "scale-specific" type of differentiation.

New Results

Theorem 1 (G., Preprint)

Let (X, d) be a complete metric space consisting of a thick family of geodesics from u to v.

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$$\sup_{y \in B_{R \cdot r_i(x)}(x)} \|f(y) - f(x) - f'(x)(\pi(y) - \pi(x))\| = o(r_i(x)) \quad \text{as } i \to \infty$$

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where π is the canonical map $X \to [0, d(u, v)]$.
The scales $r_i(x)$ can be chosen so that the fiber $\pi^{-1}(\pi(x_0))$ contains points y_i with $d(y_i, x) \sim r_i(x)$ for infinitely many i for a μ-positive set of x (this is enough to run the argument for nonembeddability).

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Corollary to Theorem 1 (G., Preprint)

$\mathbb{H} \times \ell^1$ does not contain a biLipschitz copy of a thick family of geodesics (apply differentiation argument to each component).

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Theorem 2 (G., Preprint)

In any nonRNP Banach space, one can find a biLipschitz copy of a thick family of geodesics that satisfies a true "general-scale" differentiation theorem, like G_∞.

Closing Question

- These differentiation methods even prove non-local biLipschitz embeddability (at least for separable metric spaces).

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- Question: Are there any nonlocal obstructions to biLipschitz embeddability into RNP spaces?
- More specifically, if every point of a complete, separable metric space has a neighborhood which biLipschitz embeds into some RNP space (which can depend on the point), does the entire metric space biLipschitz embed into some RNP space?

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