NonBilipschitz Embeddability into RNP Spaces: Thick Families of Geodesics and Differentiation

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## Overview of Talk

### Background
- Banach spaces having the Radon-Nikodým property (RNP space).
- Differentiation based proofs of non-RNP biLipschitz embeddability of metric measure spaces.
- Thick families of geodesics and metric characterization of RNP.

### New Results
- A “scale-specific” type of RNP differentiation on metric spaces containing a thick families of geodesics.
- An application to non-biLipschitz embeddability.
- Embedding spaces with true RNP Lipschitz differentiable structure into nonRNP spaces.
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**Examples**

- $\mathbb{R}$ - by Lebesgue’s fundamental theorem. As a corollary - all finite dimensional normed spaces.
- Reflexive spaces, such as $\ell^p$, $L^p$, $1 < p < \infty$.
- Separable dual spaces, such as $\ell^1 = c_0^*$. 
### Background: RNP spaces

#### Definition (Radon-Nikodým property)

A Banach space $V$ has the **Radon-Nikodým property** (RNP) if every Lipschitz map $\mathbb{R} \to V$ is differentiable Lebesgue-almost everywhere. In this case we call $V$ an **RNP space**.

#### Examples

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#### Nonexamples

- $L^1$, $t \mapsto \chi_{[0,t]}$, and $c_0$, $t \mapsto (\sin(nt)/n)_{n=1}^{\infty}$ are nowhere differentiable Lipschitz maps.
Laakso-Lang-Plaut Infinite Diamond Graph, $G_{\infty}$.

**Theorem (Cheeger-Kleiner ’09)**

1. For any RNP space $V$ and Lipschitz map $f : G_{\infty} \to V$, $f$ is differentiable wrt $\pi_0 : G_{\infty} \to G_0 \subseteq \mathbb{R}$ at $\mu_{\infty}$-a.e. $x \in G_{\infty}$, meaning there exists $f'(x) \in \mathbb{R}$ such that

$$f(y) - f(x) = f'(x)(\pi_0(y) - \pi_0(x)) + o(d(y, x)) \quad \text{as } y \to x$$

2. Consequently, $G_{\infty}$ does not biLipschitz embed into any RNP space.
Proof of (2).

Suppose $V$ is an RNP space and $f: G_\infty \to V$ is Lipschitz. Pick a point $x$ that is a limit point of $\pi_0^{-1}(\pi_0(x))$ and for which (1) holds.

$f(y) - f(x) = f'(x)(\pi_0(y) - \pi_0(x)) + o(d(y, x))$ as $y \to x$.

Which implies $f$ is not biLipschitz! □
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$$f(y) - f(x) = f'(x)(\pi_0(y) - \pi_0(x)) + o(d(y, x)) \quad \text{as } y \to x$$

$$f(y) - f(x) = 0 + o(d(y, x)) \quad \text{as } \pi_0^{-1}(\pi_0(x)) \ni y \to x$$

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The RNP is preserved under linear isomorphically embeddings. A popular line of research in Banach space geometry is to find purely metric characterizations of such properties.
Background: Thick Families of Geodesics

The RNP is preserved under linear isomorphic embeddings. A popular line of research in Banach space geometry is to find purely metric characterizations of such properties.

- Superreflexivity (Bourgain ’86)
- Rademacher cotype $q$ (Mendel-Naor ’08)
- Uniform $p$-convexity (Mendel-Naor ’13)
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**Theorem (Ostrovskii ’14a)**

A Banach space $V$ does not have the RNP if and only if it contains a biLipschitz copy of a thick family of geodesics.

Proof of $\iff$ is very natural, and does not use differentiation theory of Cheeger-Kleiner ’09. Ostrovskii directly constructs an $L_\infty$-bounded, $L^1$-divergent martingale, which is equivalent to nonRNP.
Definition (Ostrovskii '14a)

Let $(X, d)$ be a metric space, $u, v \in X$, and $\Gamma$ a family of geodesics connecting $u$ to $v$. $\Gamma$ is thick if there is an $\alpha > 0$ such that

\[ \text{for any } \gamma \in \Gamma \text{ and points } 0 = t_0 < t_1 < ... < t_k = d(u, v), \]

there is a superset of points $0 = t'_0 < t'_1 < ... < t'_{k'} = d(u, v)$ and another geodesic $\tilde{\gamma} \in \Gamma$ with $\gamma(t'_i) = \tilde{\gamma}(t'_i)$ and

\[ \sum_{i=1}^{k'} \max_{t \in [t'_i - 1, t'_i]} d(\gamma(t), \tilde{\gamma}(t)) \geq \alpha. \]
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Example 1:

\[ 
\begin{align*}
G_0 & \rightarrow G_1 \\
& \quad \pi_0 \\
& \quad \pi_1 \\
& \quad \pi_2 \\
& \cdots
\end{align*}
\]
Example 1:

\[ G_0 \rightarrow G_1 \rightarrow G_2 \rightarrow G_\infty \]

\[ \pi_0 \rightarrow \pi_1 \rightarrow \pi_2 \rightarrow \cdots \]

Example 2:
Non-quasiconvex deformation of diamond graphs. Replace edges with increasingly cuspidated diamonds.
Background: Thick Families of Geodesics

- Natural followup question to Ostrovskii’s theorem: Does the biLipschitz containment of a thick family of geodesics characterize geodesic metric spaces nonembeddable into RNP spaces, like it does for Banach spaces?

- No, Heisenberg group ($H$) is a counterexample (Ostrovskii '14b).

- $H$ equipped with Carnot-Caratheodory metric does not biLipschitz embed into any RNP space (Cheeger-Kleiner '06, Lee-Naor '05, Semmes '96, Pansu '89).

- For $p > 0$, call a metric space $(X, d)$ $p$-convex if there is a quasimetric $\rho$ equivalent to $d$ satisfying

$$\rho(w, y)^{p/2} + \rho(z, y)^{p/2} + \rho(x, y)^p - \left(\rho(w, x)^{p/2} - \rho(z, x)^{p/2}\right)^p \gtrsim \rho(w, z)^p$$

for all $w, x, y, z \in X$.

- $H$ is 8-convex (Li '14). Later, $H$ is 4-convex (Li '16).

- $p$-convex metric spaces do not contain biLipschitz copies of thick families of geodesics (Ostrovskii '14b).
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Motivating Question

- The proof that $\mathbb{H}$ and the original proof that $G_\infty$ do not embed into any RNP space uses a differentiation method.

- $\mathbb{H}$ does not contain a biLipschitz copy of thick family of geodesics, but do thick families of geodesics satisfy a differentiation theorem?
Motivating Question

The proof that $\mathbb{H}$ and the original proof that $G_\infty$ do not embed into any RNP space uses a differentiation method.

$\mathbb{H}$ does not contain a biLipschitz copy of thick family of geodesics, but do thick families of geodesics satisfy a differentiation theorem?

Yes, but “scale-specific” type of differentiation.
Theorem 1 (G., Preprint)

Let \((X, d)\) be a complete metric space consisting of a thick family of geodesics from \(u\) to \(v\).
New Results

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Let \((X, d)\) be a complete metric space consisting of a thick family of geodesics from \(u\) to \(v\). Then there is a compact subset \(Y \subseteq X\), a Borel probability measure \(\mu\) on \(Y\), and a sequence of scales \(r_i(x) \downarrow 0\) for each \(x \in Y\) such that for any RNP space \(V\), Lipschitz \(f: Y \rightarrow V\), and \(\mu\)-a.e. \(x \in Y\), \(\exists! f'(x) \in V\) such that for any \(R \geq 1\), \[
\sup_{y \in B_{R \cdot r_i(x)}(x)} \|f(y) - f(x) - f'(x)(\pi(y) - \pi(x))\| = o(r_i(x)) \quad \text{as} \quad i \to \infty
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where \(\pi\) is the canonical map \(X \rightarrow [0, d(u, v)]\).

The scales \(r_i(x)\) can be chosen so that the fiber \(\pi^{-1}(\pi(x_0))\) contains points \(y_i\) with \(d(y_i, x) \sim r_i(x)\) for infinitely many \(i\) for a \(\mu\)-positive set of \(x\) (this is enough to run the argument for nonembeddability).
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New Results

- If a metric space embeds into an RNP space (Ost ’14a) or is $p$-convex (Ost ’14b), then it does not contain a biLipschitz copy of a thick family of geodesics.

H does not embed into any RNP space, and $\ell_1$ is not $p$-convex, so $H \times \ell_1$ satisfies neither hypothesis.

Corollary to Theorem 1 (G., Preprint)

$H \times \ell_1$ does not contain a biLipschitz copy of a thick family of geodesics (apply differentiation argument to each component).

Theorem 2 (G., Preprint)

In any nonRNP Banach space, one can find a biLipschitz copy of a thick family of geodesics that satisfies a true “general-scale” differentiation theorem, like $G_\infty$. 

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\( \mathbb{H} \times \ell^1 \) does not contain a biLipschitz copy of a thick family of geodesics (apply differentiation argument to each component).

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In any nonRNP Banach space, one can find a biLipschitz copy of a thick family of geodesics that satisfies a true “general-scale” differentiation theorem, like \( G_\infty \).
These differentiation methods even prove non-local biLipschitz embeddability (at least for separable metric spaces).
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Question: Are there any nonlocal obstructions to biLipschitz embeddability into RNP spaces?

More specifically, if every point of a complete, separable metric space has a neighborhood which biLipschitz embeds into some RNP space (which can depend on the point), does the entire metric space biLipschitz embed into some RNP space?
References

- Bourgain '86 *The metrical interpretation of superreflexivity in Banach spaces.*
- Cheeger-Kleiner '06 *On the differentiability of Lipschitz maps from metric measure spaces to Banach spaces.*
- Cheeger-Kleiner '09 *Differentiability of Lipschitz Maps from Metric Measure Spaces to Banach Spaces with the Radon–Nikodym Property.*
- Gartland Preprint *Thick Families of Geodesics and Differentiation.*
- Lee-Naor '05 *$L_p$ metrics on the Heisenberg group and the Goemans-Linial conjecture.*
- Li '14 *Coarse differentiation and quantitative nonembeddability for Carnot groups.*
- Li '16 *Markov convexity and nonembeddability of the Heisenberg group.*
References

- Mendel-Naor '08 *Metric Cotype.*
- Mendel-Naor '13 *Markov convexity and local rigidity of distorted metrics.*
- Pansu '89 *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un.*
- Ostrovskii '14a *Radon-Nikóým property and thick families of geodesics.*
- Ostrovskii '14b *Metric spaces nonembeddable into Banach spaces with the Radon-Nikodým property and thick families of geodesics.*
- Semmes '96 *On the nonexistence of bi-Lipschitz parameterizations and geometric problems about $A_\infty$-weights.*