Lipschitz Free Spaces over Locally Compact Metric Spaces

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Motivating Questions and Overview of Talk

X = complete, locally compact metric space.

Q1 If every compact subset of X biLipschitz embeds into an RNP space, is the same true of X?

Q2 If $\mathrm{LF}(K)$ has the RNP for every $K \subseteq X$ compact, is the same true of $\mathrm{LF}(X)$?

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In the talk...

- Background on non-biLipschitz embeddability of metric spaces into RNP spaces via differentiation.
- Proof of following theorem

Theorem (G '20)

$\mathrm{LF}(X)$ *has the approximation/Schur property if* $\mathrm{LF}(K)$ *has the approximation/Schur property for every* $K \subseteq X$ *compact.*

- Partial progress towards Q2 (with RNP).

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They also proved same thing for

- weak sequential completeness
- ℓ^1-saturation

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Examples and Nonexamples

- ✓ Reflexive spaces, such as ℓ^p, L^p, $1 < p < \infty$.
- ✓ Separable dual spaces, such as $\ell^1 = c_0^*$.
- X L^1, $t \mapsto \chi_{[0,t]}$ is nowhere differentiable.
- X c_0, $t \mapsto (\sin(nt)/n)_{n=1}^{\infty}$ is nowhere differentiable.

Differentiation on metric spaces with an aim towards nonembeddability

Differentiation with Respect to a Structure Map

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We say that f is **differentiable with respect to ψ** at $p \in X$ if there exists $T : \mathbb{R}^k \to V$ linear such that

$$f(p) - f(x) = T(\psi(p) - \psi(x)) + o(d(p, x)) \quad \text{as} \quad x \to p$$

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When $X = \mathbb{R}^k$, V has RNP, and $\psi = \mathrm{id}_{\mathbb{R}^k}$, **Rademacher's theorem** says that f is differentiable wrt ψ Lebesgue-a.e.

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Fix a structure map $\psi : X \to \mathbb{R}^k$ and Banach space V.

Observation 1 (Cheeger '99)

If, for every Lipschitz $f : X \to V$,

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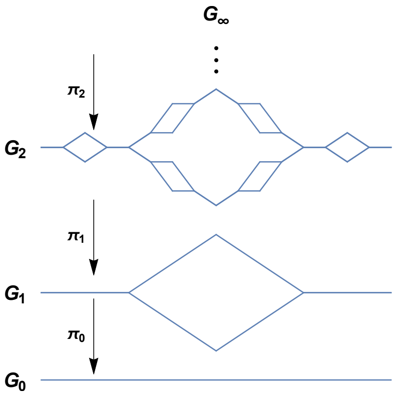
contradiction. □

Two Examples

Laakso Space



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Laakso space, G_∞, is the metric inverse limit of the system $G_0 \overset{\pi_0}{\leftarrow} G_1 \overset{\pi_1}{\leftarrow} G_2 \overset{\pi_2}{\leftarrow} \dots$

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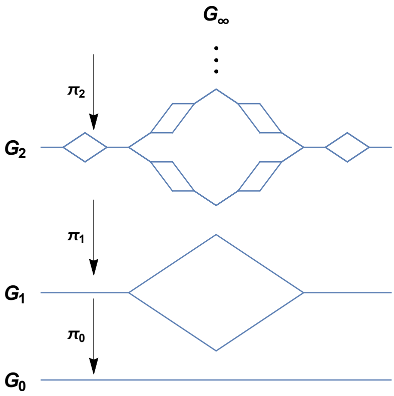

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Theorem (Cheeger-Kleiner '09)

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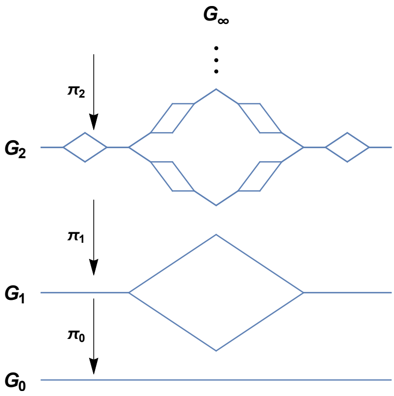

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Different proof of nonembeddability by Ostrovskii in '11.

The Heisenberg Group

Definition (Heisenberg Group)

- $\mathbb{H} = (\mathbb{R}^3, (x, y, t) * (x', y', t') = (x + x', y + y', t + t' - 2xy' + 2yx'))$

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- $N(g*h) \leq N(g) + N(h)$

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Theorem (Pansu '89, Semmes '96, C-K '06, Lee-Naor '06)

If $f : \mathbb{H} \to V$ is Lipschitz and V has RNP, then f is differentiable wrt $\pi^{ab} : \mathbb{H} \to \mathbb{H}^{ab} = \mathbb{R}^2$ a.e. Consequently, $\mathbb{H}$ does not embed into V by Observation 1.

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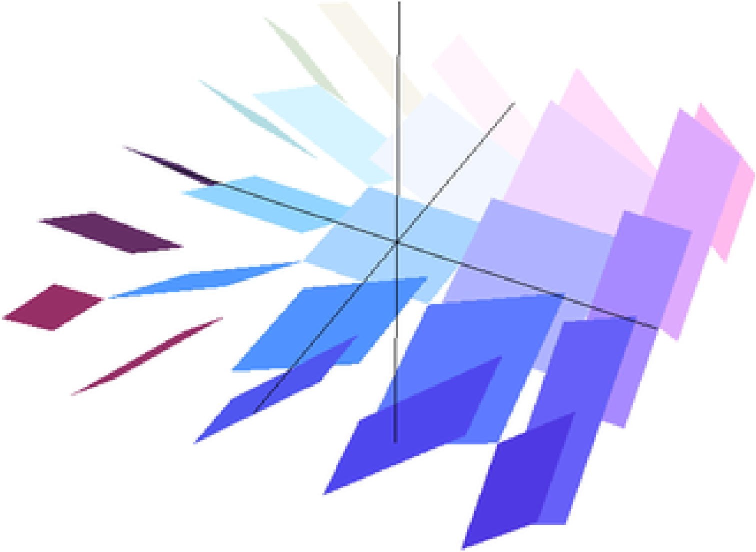

http://www.asanchezmath.com/heisenberg-renders.html

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  - Uniformly discrete (Kalton '04).
  - Snowflake of proper metric or separable dual Banach space (Kalton '04).
  - Proper and countable (Dalet '15).
  - Ultrametric (Cúth-Doucha '15).
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Q2 If $\mathrm{LF}(K)$ has the RNP for every $K \subseteq X$ compact, is the same true of $\mathrm{LF}(X)$?

Before discussing Q2, quick review of Lipschitz free spaces.

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- Weak* topology on $B_{\mathrm{Lip}_0(X)}$ is topology of pointwise convergence.
- When $Y \subseteq X$, $\mathrm{LF}(Y) \subseteq \mathrm{LF}(X)$ canonically.

Back to Q2: Easier to answer if RNP is replaced by AP or SP

Local to Global Properties of Lipschitz Free Spaces

- V has the **approximation property** (AP) if $\mathcal{B}(V) = \overline{\mathcal{F}(V)}^{\mathrm{ucc}}$, where ucc = uniform convergence on compacta.
- V has the **Schur property** (SP) if every weakly null sequence is norm null.

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Proof. Use previous theorem and Lancien-Pernecká '13: $\mathrm{LF}(X)$ has (B)AP for every doubling X ($\exists N, B_r(p) \subseteq \cup_{i=1}^{N} B_{r/2}(p_i)$).

Revisiting local compactness with an aim towards the proof of Theorem 1

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  - $\mathrm{lm}(\mathrm{lm}(\mathcal{D})) = \mathrm{lm}(\mathcal{D})$.
  - $\mathrm{ulm}(\mathrm{ulm}(\mathcal{D})) \neq \mathrm{ulm}(\mathcal{D})$, what does $\mathcal{D}$ generate under repeated applications of ulm?

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$\mathcal{D}'$ is **ulm-closed** if $\mathrm{ulm}(\mathcal{D}') = \mathcal{D}'$.

Local Modeling of Subsets

- X = complete metric space, $\mathcal{D}$ = downward-closed collection of closed subsets of X (for example, $\mathcal{D}$ = compacts).

$F \subseteq X$ is **locally modeled on $\mathcal{D}$** if $\forall p \in X$, $\exists \theta > 0$, $F \cap B_\theta(p) \in \mathcal{D}$,
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ulm-closure of $\mathcal{D}$ is smallest ulm-closed, downward-closed collection containing $\mathcal{D}$.

Proof that AP is Local-to-Global

Lemma (1)

If $\mathcal{I} \subseteq \mathcal{P}(X)$ is an ideal, then the ulm-closure of $\mathcal{I}$ is lm($\mathcal{I}$).

Proof that AP is Local-to-Global

Lemma (1)

If $\mathcal{I} \subseteq \mathcal{P}(X)$ is an ideal, then the ulm-closure of $\mathcal{I}$ is Im($\mathcal{I}$).

Lemma (2)

Assume X is bndd. $\mathcal{D}_1 := \{F \textit{ clsd} : \mathcal{B}(\mathrm{LF}(X), \mathrm{LF}(F)) \subseteq \overline{\mathcal{F}(\mathrm{LF}(X))}^{\mathrm{ucc}}\}$.
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Theorem 1 (G '20)

Let $\mathcal{I} \subseteq \mathcal{P}(X)$ be an ideal. If $X \in \mathrm{Im}(\mathcal{I})$ and $\mathrm{LF}(A)$ has AP/SP for every $A \in \mathcal{I}$, then $\mathrm{LF}(X)$ has AP/SP.

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Partial Progress on RNP Local-to-Global

X = countable, discrete metric space, basepoint $0 \in X$.

Definition

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Theorem 2 (G '20)

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Corollary

If $X \in Fin_n$ for some n, then $\mathrm{LF}\,(X) = \mathrm{LF}\,(n, X)$ has RNP.

Thanks!

Questions?

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