The Ribe Program and Markov Convexity of Filiform Groups

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Overview

- Banach space background
    - Finite representability of Banach spaces, local properties, Ribe's theorem
    - Examples in Ribe program: metric cotype, Markov convexity
- Markov convexity of filiform groups
    - Background on Carnot and filiform groups
    - Known and new results on their Markov convexity
    - Proof ideas
- Generalizing to other groups: a combinatorial problem

Banach Space Background: Finite Representability

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Examples

- If V and W are isomorphic, then V and W are mutually finitely representable.
- ℓ^p and L^p are mutually finitely representable, but not isomorphic if $p \neq 2, \infty$.

Banach Space Background: Local Properties

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A property of Banach spaces is **local** if it is preserved under mutual finite representability.

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- For a parallelpiped in V spanned by finitely many vectors,
  - V has **type $\boldsymbol{p}$** if $\sum \|\text{diagonals}\|^p \lesssim \sum \|\text{sides}\|^p$.
  - V has **cotype $\boldsymbol{q}$** if $\sum \|\text{diagonals}\|^q \gtrsim \sum \|\text{sides}\|^q$.
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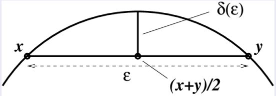
  - L^p has type maximal $\min(2, p)$ (for $p < \infty$) and minimal cotype $\max(2, p)$ (for any p).
- V is $\boldsymbol{p}$**-convex** if it admits an equivalent norm with $\delta(\epsilon) \gtrsim \epsilon^p$.
- L^p is (minimally) $\max(2, p)$-convex if $1 < p < \infty$.



Banach Space Background: Ribe Program

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If V and W are uniformly homeomorphic, then V and W are mutually finitely representable.

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Definition: Bourgain '86

The **Ribe program** is the research program concerned with finding metric reformulations of local properties of Banach spaces.

Ribe Program: Metric Cotype

Example: Metric Cotype, Mendel-Naor '05

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- Väisälä '96: What about $L^p \overset{QS}{\hookrightarrow} L^q$ when $\max(q, 2) < p$?
- Naor used metric cotype to answer this question negatively in '06.

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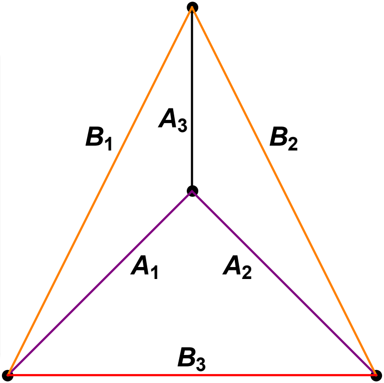
A metric space is **Markov p-convex** if it admits an equivalent quasi-metric satisfying

$$A_3^p + \frac{A_1^p}{2} + \frac{A_2^p}{2} - \frac{{B_1}^p}{2^p} - \frac{{B_2}^p}{2^p} \gtrsim {B_3}^p$$



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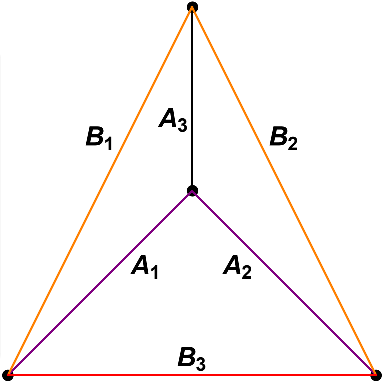

Disclaimer: Our definition of Markov p-convexity is not the one used by Lee-Naor-Peres or Mendel-Naor.

Markov Convexity of Carnot Groups

Theorem: Li '13, '14

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Theorem 1

- Every Carnot group of step s is Markov p-convex for every $p \geq 2s$.
- The filiform group of step s is Markov p-convex if and only if $p \geq 2s$.

Background: Carnot Groups

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- A Lie algebra $\mathfrak{g}$ is **stratified** of **step** s if it admits a decomposition $\mathfrak{g} = \oplus_{i=1}^{s} \mathfrak{g}_i$ with $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$ and $\mathfrak{g}_s \neq 0$.

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Example: Filiform Groups

- The **filiform group** of **step** $\boldsymbol{s}$ (F_s) is the Carnot group with Lie algebra $\mathfrak{g} = \oplus_{i=1}^{s} \mathfrak{g}_i$
  - $\mathfrak{g}_1 = \text{span}(X, Y_1)$,
  - $\mathfrak{g}_i = \text{span}(Y_i)$, $2 \leq i \leq s$,
  - $[X, Y_i] = Y_{i+1}$, all other brackets $= 0$.
- F_2 is the Heisenberg group, F_3 is the Engel Group.

Background: Carnot Groups: Homogen. Metrics

Definition: Homogeneous Metric

- For any Carnot group G, there are dilations $\{\delta_t\}_{t>0} \subseteq \mathrm{Aut}(G)$ with $\delta_t(\exp(X)) = \exp(tX_1 + t^2 X_2 + \dots t^s X_s)$.

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- A **homogeneous metric** on G is a metric d satisfying
  - $d(gh_1, gh_2) = d(h_1, h_2)$ (left-invariance)
  - $d(\delta_t(h_1), \delta_t(h_2)) = td(h_1, h_2)$ (homogeneity)
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Example

- $N(\exp(X)) := \lambda_1\|X_1\| + \lambda_2\|X_2\|^{1/2} + \dots \lambda_s\|X_s\|^{1/s}$ for some $\lambda_i > 0$
- $d(g, h) := N(h^{-1}g)$

Carnot Groups: Non-BiLipschitz Embeddability

Theorem: Pansu '89, Semmes '96

Let G, H be two Carnot groups. There is a biLipschitz embedding of G into H if and only if there is a graded Lie algebra embedding from $\mathfrak{g}$ into $\mathfrak{h}$.

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Theorem 1 again

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Theorem 1 again

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- Can also use to prove nonexistence of Lipschitz subquotient maps and quantitative non-biLipschitz embeddability.

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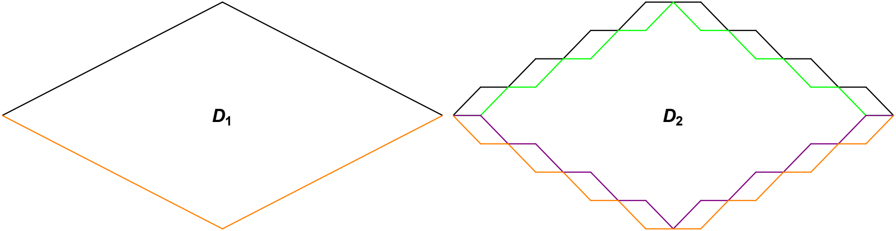
- By Mendel-Naor, suffices to map diamond graphs D_m into $\mathfrak{f}_s$ via ϕ_m (and then into F_s via exp) with "good" control on Lipschitz constant and $m^{\frac{1}{2s}+\epsilon}$ bound on coLipschitz constant between "vertical pairs".



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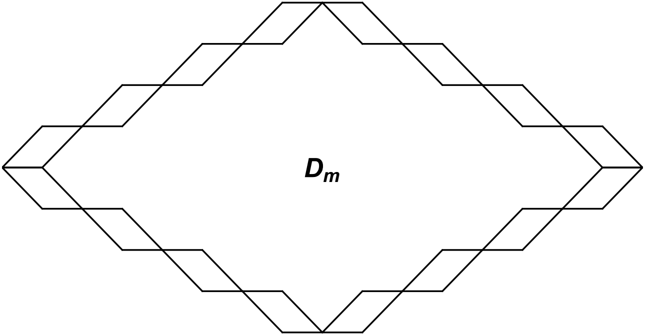

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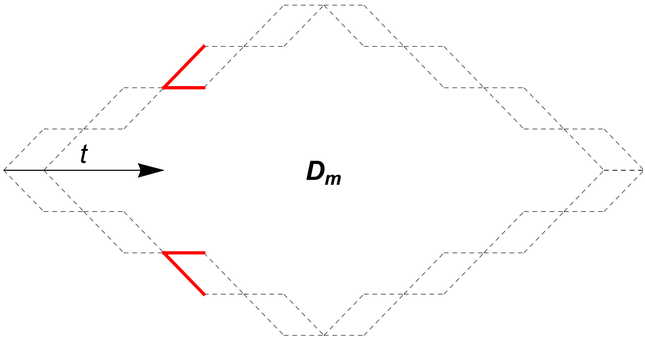

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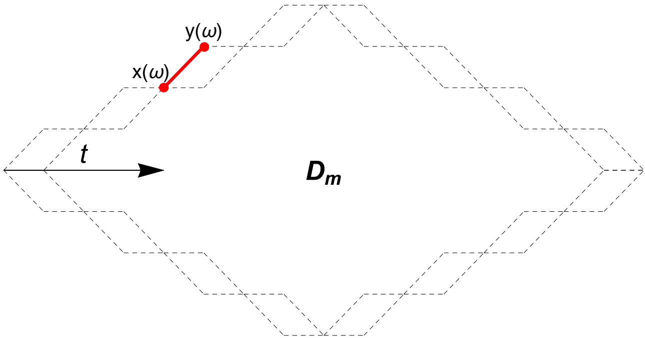

Choose m ($m = 2$ in pic). Choose t ($t = 4$ in pic). Choose $\omega \in \{\text{Heads}, \text{Tails}\}^m$. Get $x(m,t,\omega), y(m,t,\omega) \in D_m$.

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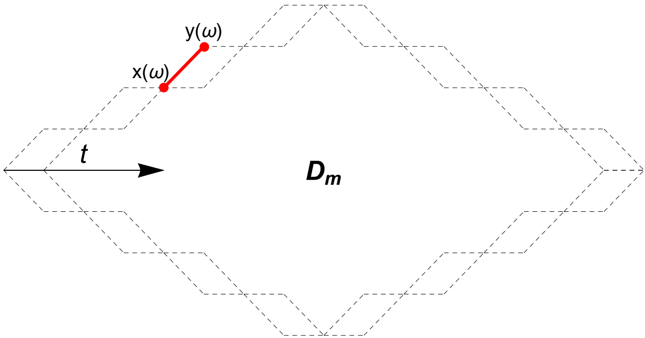
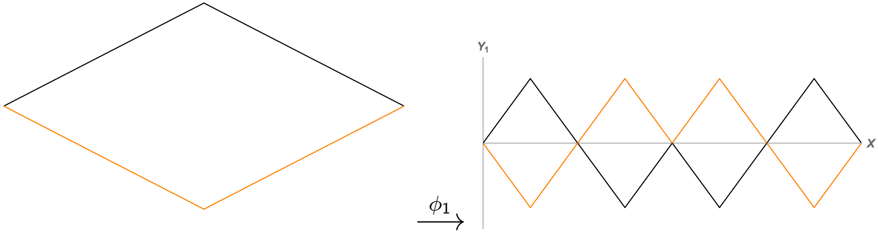


Choose m ($m = 2$ in pic). Choose t ($t = 4$ in pic). Choose $\omega \in \{\text{Heads}, \text{Tails}\}^m$. Get $x(m,t,\omega), y(m,t,\omega) \in D_m$. Need $\mathbb{E}_\omega[d(\phi_m(x(m,t,\omega)), \phi_m(y(m,t,\omega)))^{2s}] \lesssim 8^{1-m}$ independent of m, t.

Mapping the Diamonds into $\mathfrak{f}_3$



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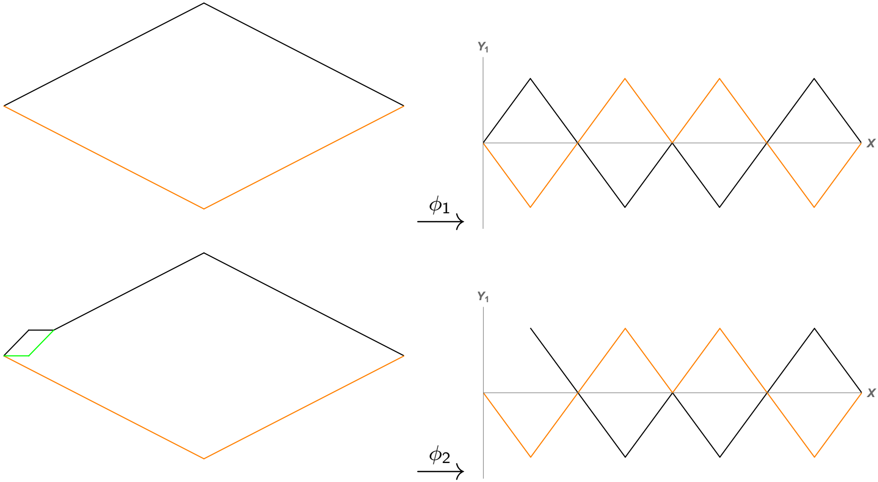

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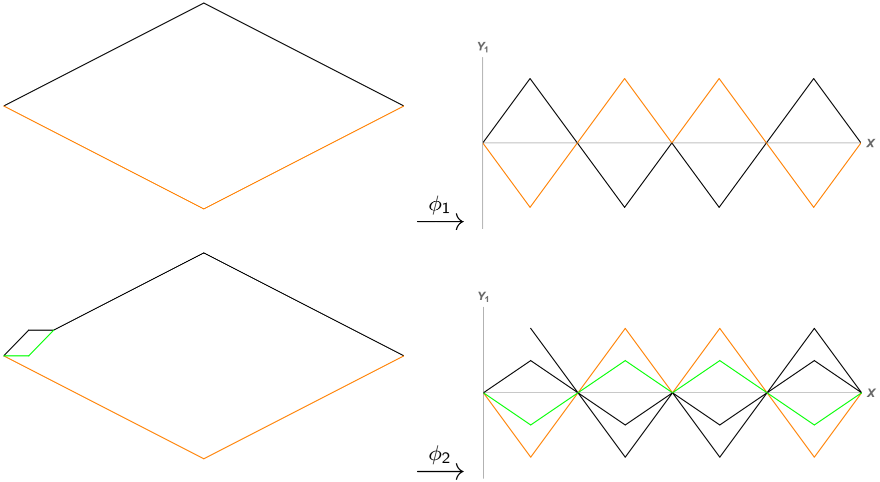

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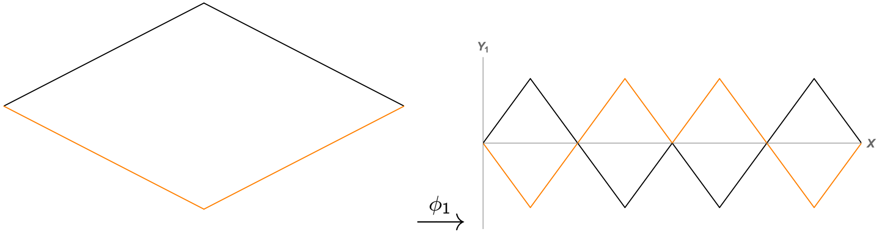

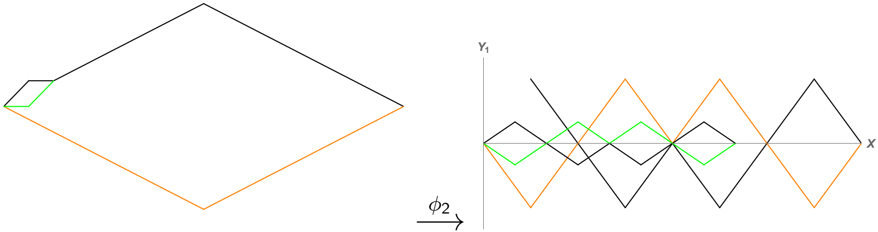

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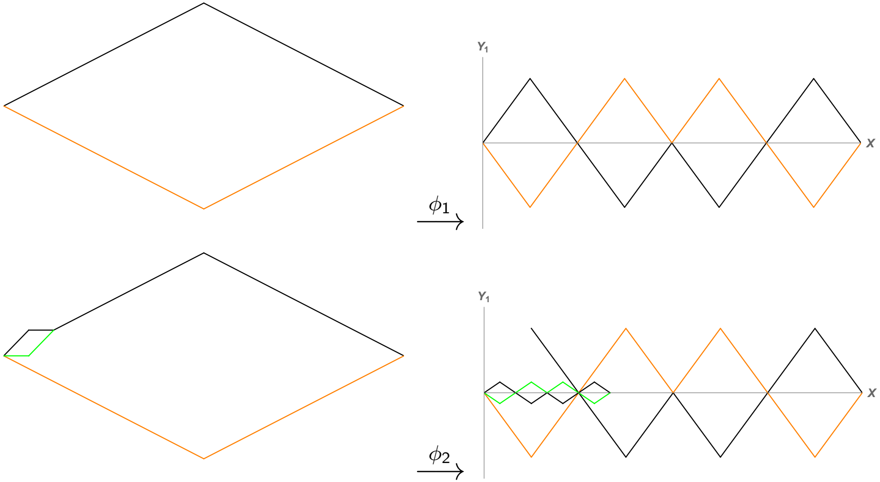

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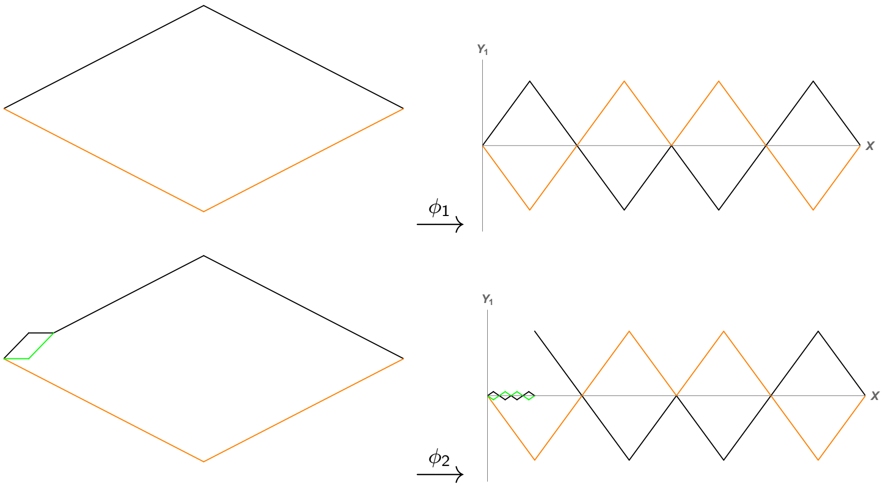

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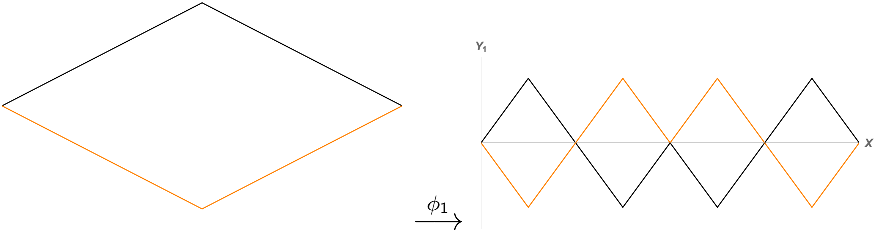

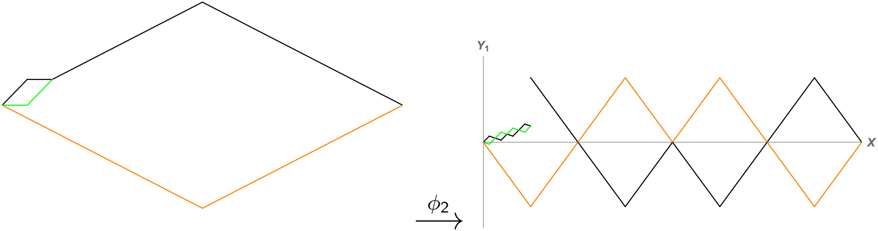

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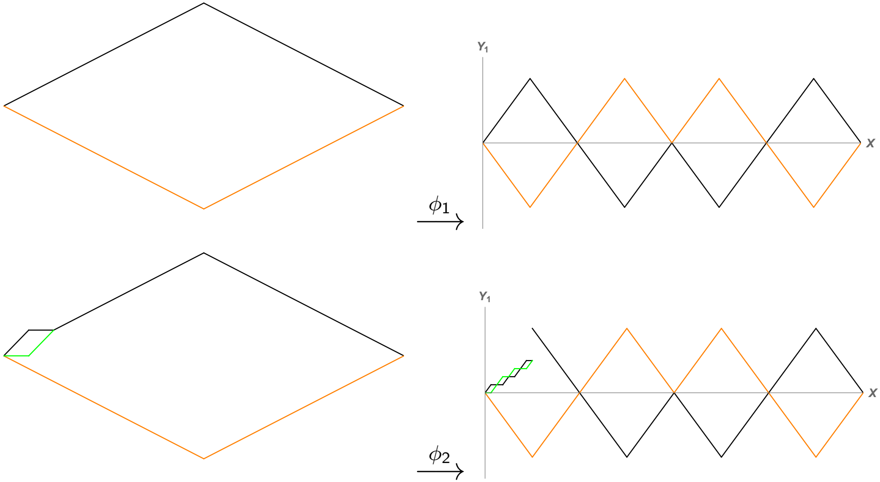

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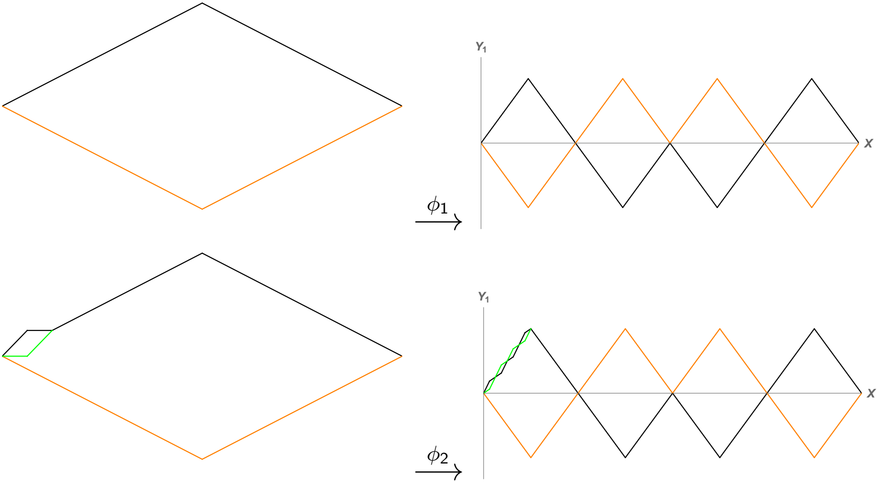

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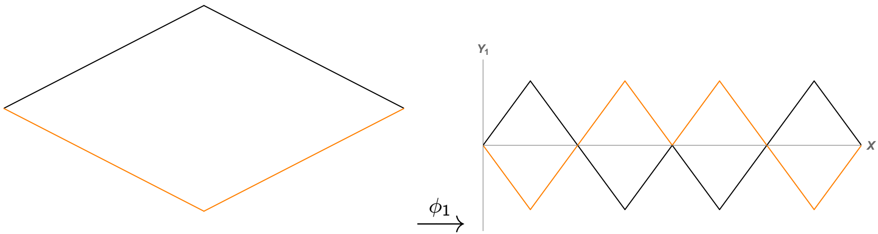

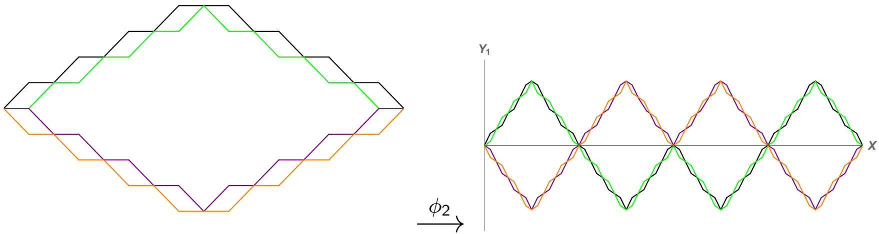

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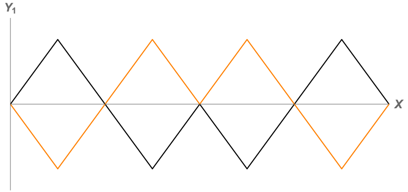

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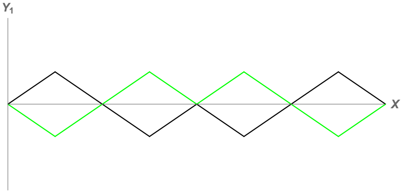

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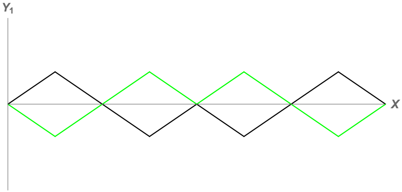

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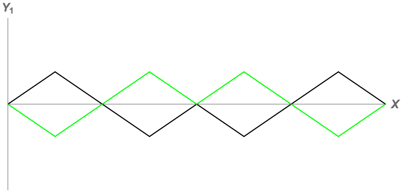

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- Then the group element joining the image of opposite vertices in smallest diamonds is $\mathrm{ad}^{(s-1)}_{8^{1-m}X}(8^{1-m}c_m Y_1) =$ $[8^{1-m}X, [8^{1-m}X, \ldots [8^{1-m}X, 8^{1-m}c_m Y_1] \ldots]] = 8^{(1-m)s}c_m Y_s$.

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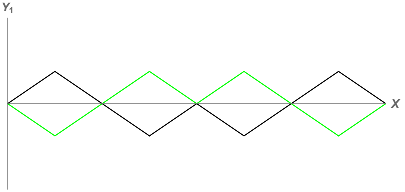

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- Hence their distance is $N(c_m Y_s) \sim (8^{(1-m)s}c_m)^{\frac{1}{s}} = 8^{1-m}m^{-\frac{1}{2s}-\frac{\epsilon}{s}}$. This mostly explains the coLipschitz bound on vertical pairs.

Generalizing to other Groups: a Combin. Problem

- A crucial property of F_s that allowed the previous argument to work is that there are $X, Y \in \mathfrak{g}_1$ with $\mathrm{ad}_X^{(s-1)}(Y) \neq 0$.
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- Must such vectors exist for every stratified $\mathfrak{g}$ of step s?
- No, but you don't see a counterexample until step 9!

$$\mathfrak{g} = \left\{ \begin{bmatrix} 0 & a & 0 & d & f & h & 0 & n & r & t \\ & 0 & a & c & e & 0 & h & k & p & s \\ & & 0 & b & 0 & e & -f & -h & \ell & q \\ & & & 0 & b & -c & d & 0 & i & m \\ & & & & 0 & a & 0 & d & g & j \\ & & & & & 0 & a & c & -2d & -g \\ & & & & & & 0 & b & -c & d \\ & & & & & & & 0 & a & 0 \\ & & & & & & & & 0 & a \\ & & & & & & & & & 0 \end{bmatrix} : a, \ldots t \in \mathbb{R} \right\}$$

Generalizing to other Groups: a Combin. Problem

- Suppose $\mathfrak{g}$ is a stratified Lie algebra of step 5.
- Let's try to prove $\exists X, Y \in \mathfrak{g}_1$ such that $[X,[X,[X,[X,Y]]]] \neq 0$.

Generalizing to other Groups: a Combin. Problem

- Suppose $\mathfrak{g}$ is a stratified Lie algebra of step 5.
- Let's try to prove $\exists X, Y \in \mathfrak{g}_1$ such that $[X,[X,[X,[X,Y]]]] \neq 0$.
- By fundamental structural theorems, we may assume $\mathfrak{g}$ is a graded subalgebra of strictly upper triangular 6×6 matrices.

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- Suppose $\mathfrak{g}$ is a stratified Lie algebra of step 5.
- Let's try to prove $\exists X, Y \in \mathfrak{g}_1$ such that $[X,[X,[X,[X,Y]]]] \neq 0$.
- By fundamental structural theorems, we may assume $\mathfrak{g}$ is a graded subalgebra of strictly upper triangular 6×6 matrices.
- since $\mathfrak{g}$ is stratified of step 5, there exists

$$\begin{bmatrix} & a & 0 & 0 & 0 & 0 \\ & & b & 0 & 0 & 0 \\ & & & c & 0 & 0 \\ & & & & d & 0 \\ & & & & & e \\ & & & & & \end{bmatrix} \in \mathfrak{g}$$

  with $a, b, c, d, e \neq 0$.
- By applying a graded isomorphism that scales each entry,

$$\begin{bmatrix} & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \\ & & & & & \end{bmatrix} \in \mathfrak{g}' \cong \mathfrak{g}$$

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$$\begin{bmatrix} a & 0 & 0 & 0 & 0 \\ & b & 0 & 0 & 0 \\ & & c & 0 & 0 \\ & & & d & 0 \\ & & & & e \end{bmatrix} \in \mathfrak{g}'$$

with not all a, b, c, d, e equal and $x \in \mathbb{R}$ be arbitrary.

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  with not all a, b, c, d, e equal and $x \in \mathbb{R}$ be arbitrary.

- Then

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} x+a & 0 & 0 & 0 & 0 \\ & x+b & 0 & 0 & 0 \\ & & x+c & 0 & 0 \\ & & & x+d & 0 \\ & & & & x+e \end{bmatrix} \in \mathfrak{g}'$$

  By applying the scaling isomorphism to each entry, we get

$$\begin{bmatrix} (x+a)^{-1} & 0 & 0 & 0 & 0 \\ & (x+b)^{-1} & 0 & 0 & 0 \\ & & (x+c)^{-1} & 0 & 0 \\ & & & (x+d)^{-1} & 0 \\ & & & & (x+e)^{-1} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{bmatrix} \in \mathfrak{g}'' \cong \mathfrak{g}'$$

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Take

$$Y = \begin{bmatrix} (x+a)^{-1} & 0 & 0 & 0 & 0 \\ & (x+b)^{-1} & 0 & 0 & 0 \\ & & (x+c)^{-1} & 0 & 0 \\ & & & (x+d)^{-1} & 0 \\ & & & & (x+e)^{-1} \end{bmatrix}, X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{bmatrix}$$

$$[X, Y] =$$

$$= \begin{bmatrix} (x+b)^{-1}-(x+a)^{-1} & 0 & 0 & 0 \\ & (x+c)^{-1}-(x+b)^{-1} & 0 & 0 \\ & & (x+d)^{-1}-(x+c)^{-1} & 0 \\ & & & (x+e)^{-1}-(x+d)^{-1} \end{bmatrix}$$

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$$[X, [X, Y]] = \begin{bmatrix} A - 2B + C & 0 & 0 \\ & B - 2C + D & 0 \\ & & C - 2D + E \\ & & \\ & & \end{bmatrix}$$

where $A = (x + a)^{-1}$, $B = (x + b)^{-1}$, etc.

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$$[X,[X,[X,Y]]] = \begin{bmatrix} & -A+3B-3C+D & 0 \\ & & -B+3C-3D+E \\ & & \\ & & \\ & & \end{bmatrix}$$

where $A = (x+a)^{-1}$, $B = (x+b)^{-1}$, etc.

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$$Y = \begin{bmatrix} (x+a)^{-1} & 0 & 0 & 0 & 0 \\ & (x+b)^{-1} & 0 & 0 & 0 \\ & & (x+c)^{-1} & 0 & 0 \\ & & & (x+d)^{-1} & 0 \\ & & & & (x+e)^{-1} \end{bmatrix}, X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{bmatrix}$$

$$[X,[X,[X,[X,Y]]]] = \begin{bmatrix} & & & & A - 4B + 6C - 4D + E \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} = 0?$$

where $A = (x+a)^{-1}$, $B = (x+b)^{-1}$, etc.

Generalizing to other Groups: a Combin. Problem

$$\forall x, (x+a)^{-1} - 4(x+b)^{-1} + 6(x+c)^{-1} - 4(x+d)^{-1} + (x+e)^{-1} = 0?$$

Is there a nontrivial, 0-sum subset of $\{1, -4, 6, -4, 1\}$?

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$\{1, -6, 15, -20, 15, -6, 1\}$?

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Is there a nontrivial, 0-sum subset of $\{1, -4, 6, -4, 1\}$? X

$\{1, -6, 15, -20, 15, -6, 1\}$? X

$\{1, -8, 28, -56, 70, -56, 28, -8, 1\}$?

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$\{1, -8, 28, -56, 70, -56, 28, -8, 1\}$? ✓

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$\{1, -6, 15, -20, 15, -6, 1\}$? ✗

$\{1, -8, 28, -56, 70, -56, 28, -8, 1\}$? ✓

Our counterexample $\mathfrak{g}$ is generated by

$$X = \begin{bmatrix} 1 &&&&&&&& \\ & 1 &&&&&&& \\ && 1 &&&&&& \\ &&& 1 &&&&& \\ &&&& 1 &&&& \\ &&&&& 1 &&& \\ &&&&&& 1 && \\ &&&&&&& 1 & \\ &&&&&&&& 1 \end{bmatrix} \quad Y = \begin{bmatrix} 0 &&&&&&&& \\ & 0 &&&&&&& \\ && 1 &&&&&& \\ &&& 1 &&&&& \\ &&&& 0 &&&& \\ &&&&& 0 &&& \\ &&&&&& 1 && \\ &&&&&&& 0 & \\ &&&&&&&& 0 \end{bmatrix}$$

Generalizing to other Groups: a Combin. Problem

$\forall x, (x+a)^{-1} - 4(x+b)^{-1} + 6(x+c)^{-1} - 4(x+d)^{-1} + (x+e)^{-1} = 0?$

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Counterexamples in steps 9,14,15,21,25,27,30,32,33,34,35,36,39,...

Thanks!

Questions?

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