Math 221, EL1 - Final Exam - December 14, 2012

Name:__________________________________________________________

Signature:_____________________________________________________

Circle your discussion section:   ED1 (9:00, Nathan)   ED2 (1:00, Dominic)
EDA (9:00, Sarka)         EDB (11:00, Sarka)     EDC (12:00, Sneha)
EDD (1:00, Sneha)       EDE (2:00, Panupong)    EDF (3:00, Panupong)

DO NOT OPEN EXAM UNTIL TOLD TO DO SO.

Time: 180 minutes. You may not use any books or calculator; you may use a single handwritten notecard. There are 180 points possible. To get any credit, you must show your work. Unless indicated, you do not need to simplify your answers. Partial credit will be based only on what is actually written on the paper. All intermediate steps should be correct as written.

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1. (4 points per part) The height of a projectile at time $t$ seconds is given by $h(t)$ meters.

(a) Write an expression for the projectile’s average velocity from time $t = 1$ to $t = 3$.

(b) Given that $h(1) = 55$ and $h(1.001) = 55.003001$, what is the average velocity from time $t = 1$ to time $t = 1.001$?

(c) Do some simplification to find a good guess for the instantaneous velocity at time $t = 1$. Do not simply copy your answer from part (b).

(d) What does all this have to do with $h'(1)$ (one sentence)?
2. Let \( c \) be the constant value that represents the speed of light. For a very fast-moving 1 kg object that is moving very fast, the *relativistic mass* is given by

\[
    f(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

Notice that the speed of light is a constant \( c \) in this equation. \( v \) represents the velocity of the object.

(a) (2 pts) If the object is moving at a speed of \( 1 \text{ m/s} \), find its relativistic mass (This question is asking you to find \( f(1) \)).

(b) (4 pts) Find \( \lim_{v \to 0} f(v) \).

(c) (4 pts) Find \( \lim_{v \to c^-} f(v) \).

(d) (6 pts) Suppose the velocity \( v \) of the object is given by \( v(t) = t^3 \). Find \( f'(t) \). No need to simplify, but everything should definitely be in terms of \( t \).
3. (4 pts per part) Find the following derivatives using any method.

(a) Find \( \frac{dy}{dx} \) if \( y = x^2 + \sin x + \ln x + e^2 \).

(b) Find \( \frac{dy}{dx} \) if \( y = \frac{20x + 10}{x + 1} \).

(c) Find \( \frac{dy}{dx} \) if \( y = -|x| \).

(d) Find \( \frac{dy}{dx} \) if \( y = e^{\sin(2x+1)} \).
4. (a) (8 pts) Find the derivative of \( g(x) = \sqrt{x} + 1 \) using the definition of the derivative. This means that you should not use the power rule and you should not use L’Hospital’s rule.

(b) (8 pts) Find \( \lim_{x \to 2} \frac{1 - \frac{2}{x}}{4 - x^2} \). Do not use L’Hospital’s rule.
5. (6 points per part) Consider the function $h(x) = \begin{cases} x^5 + x + 1, & \text{if } x \leq 1 \\ -x^7 - x + 5, & \text{if } x > 1 \end{cases}$

(a) The function $h(x)$ is continuous. Show why.

(b) Use the Intermediate Value Theorem to show that there exists some $c$ where $h(c) = 0$.

(c) What is wrong with the following argument? Hint: The derivative is calculated correctly; something else is wrong.

We want to show that $h(x)$ has exactly one real root. We already showed that there is one real root at $x = c$ in the previous part. Let’s see what happens if we assume there is another root at $x = d$. In that case, there would be two x-values where $h(x) = 0$. This means we can use Rolle’s Theorem to show that there is at least one point where $h'(x) = 0$.

But the derivative of the function is $h'(x) = \begin{cases} 5x^4 + 1, & \text{if } x < 1 \\ \text{undefined}, & \text{if } x = 1. \\ -7x^6 - 1, & \text{if } x > 1 \end{cases}$

The derivative of this function is always positive for $x < 1$ and always negative for $x > 1$, so $h'(x)$ is never 0. This means that our assumption was wrong and there is only one real root after all.

What was wrong with that argument?
6. (6 points per part) A sample of radioactive material starts \((t = 0)\) with a mass of 10 kg. After one day \((t = 1)\), only 8 kg remain due to exponential decay.

(a) Write an equation for the mass \(M(t)\) that is remaining after \(t\) days.

(b) What is the half-life of this material?
7. (10 points) A square is shrinking at a rate of 3 \( \frac{m^2}{s} \). How fast is the side length shrinking when the square is 4 m by 4 m?

8. (10 points) While working with electric charges inside a cell, you find yourself constantly plugging very small numbers into the equation:

\[
E(r) = \frac{1}{3}e^{-4r}
\]

To simplify things, you decide to replace \( E(r) \) with a linearization \( L(r) \), based at \( r = 0 \). What is the equation of this linearization?
9. (12 points) Find the intervals where the function \( f(x) = 12x^5 - 20x^4 + 30x \) is concave up and concave down. Find all its inflection points.

10. (10 points) A model used for the yield \( Y \) of an agricultural crop as a function of the nitrogen level \( N \) in the soil is

\[
Y = \frac{kN}{1+N^2}
\]

where \( k \) is a positive constant. Given that we only use this model for values of \( N \) between 0 and 4, what nitrogen level gives the best yield?
11. (10 points) Find the area bounded by the graphs of \( y = e^x, \ y = 1, \) and \( x = 4. \)

12. (10 points) Set up and evaluate an integral to find the volume of a cone with radius 3 and height 3, using either cross-sectional areas, disks, or shells (your choice). Make sure to show where the integral comes from!
13. (4 points per part) Evaluate the following integrals using any method.

(a) \[ \int 2x + \cos x + \frac{1}{x} + e^2 \, dx \]

(b) \[ \int_0^4 \sqrt{16 - x^2} \, dx \]

(c) \[ \int \frac{(\sqrt{x} + 1)^{10}}{x + x^{2/3}} \, dx \]
14. (4 points per part) A huge spring has natural length 4 meters. Remember that Hooke’s Law for springs is $F = kx$, where $k$ is the spring constant and $x$ is the distance stretched beyond the natural length.

(a) If it takes 20 N of force to hold the spring at a length of 6 meters, what is the spring constant?

(b) Write an integral to find the work done in stretching the spring from 6 m to 9 m.

(c) While the spring is being stretched from 6 m to 9 m, the force is changing continuously. Write an integral that finds the average value of the force during this action. Then find that average value.