1. Compute the values below.

(a) (4 points) \( \int_{-2}^{1} \frac{1}{x^2} \, dx \)

We can write \( \frac{1}{x^2} \) as \( x^{-2} \), so an antiderivative looks like \( -x^{-1} \). Thus we get 
\[ -\frac{1}{x} \Big|_{-2}^{1} = -\frac{1}{2} - (-\frac{1}{-2}) = \frac{1}{2}. \]

(b) (4 points) \( \int \sec^2 x + \cos x + \frac{1}{x} + \sqrt{x} \, dx \)

Each of these pieces has an antiderivative that we can memorize. Since it is an indefinite integral we need to write +C at the end.
\[ \tan x + \sin x + \ln |x| + \frac{2}{3}x^{3/2} + C \]

(c) (4 points) \( \frac{d}{dx} \int_{x^2}^{2x} \sin^2(t^2 + 7) \, dt \)

This is an application of the Fundamental Theorem of Calculus Part I.
\[ \sin^2((2x)^2 + 7) \cdot 2 - \sin^2(x^2 + 7). \]

2. Compute the integrals below.

(a) (4 points) \( \int \frac{\sin(\sqrt{x})}{\sqrt{x}} \, dx \)

Whenever the problem with an integral is a function inside another one, we should use substitution. Here \( u = \sqrt{x} \) works. Then \( du = \frac{1}{2\sqrt{x}} \, dx \).
\[ \int 2 \sin u \, du = -2 \cos u + C = -2 \cos(\sqrt{x}) + C. \]

(b) (4 points) \( \int_{0}^{2} x^5 \sqrt{x^3 + 1} \, dx \)

Again the problem is a function inside another one, so we use substitution: \( u = x^3 + 1, du = 3x^2 \, dx \). From there we get
\[ \int_{1}^{9} \frac{1}{3} \sqrt{u} \, du. \]

We still have an \( x^3 \) in the integral, so we need to convert that to \( u \) somehow. Luckily \( x^3 = u - 1 \), so we are fine:
\[
\int_{1}^{9} x^3 \frac{1}{3} \sqrt{u} \, du = \int_{1}^{9} (u - 1) \frac{1}{3} \sqrt{u} \, du
\]
\[
= \frac{1}{3} \int_{1}^{9} u \sqrt{u} - \sqrt{u} \, du
\]
\[
= \frac{1}{3} \left( \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) \Bigg|_{1}^{9}
\]
\[
= \frac{1}{3} \left( \frac{2}{5} 9^{5/2} - \frac{2}{3} 9^{3/2} - \frac{2}{5} 1^{5/2} + \frac{2}{3} 1^{3/2} \right)
\]
\[
= \frac{1}{3} \left( \frac{2}{5} 243 - \frac{2}{3} 27 - \frac{2}{5} + \frac{2}{3} \right)
\]
\[
= 1192/45.
\]
(c) (4 points) \[ \int_1^4 \frac{x + x^2 + \sqrt{x}}{\sqrt{x}} \, dx \]

There are two ways to do this; you could try a substitution \( u = \sqrt{x} \). I prefer the method of simplifying first, then integrating:

\[
\int_1^4 \frac{x + x^2 + \sqrt{x}}{\sqrt{x}} \, dx = \int_1^4 \sqrt{x} + x^{3/2} + 1 \, dx
\]

\[
= \left. \frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2} + x \right|_1^4
\]

\[
= \frac{2}{3}4^{3/2} + \frac{2}{5}4^{5/2} + 4 - \frac{2}{3}1^{3/2} - \frac{2}{5}1^{5/2} - 1
\]

\[
= \frac{16}{3} + \frac{64}{5} + 4 - \frac{2}{3} - \frac{2}{5} - 1
\]

\[
= \frac{301}{15}.
\]

3. Consider the function \( f(x) = \sqrt{1 - x^2} \) between \( x = 0 \) and \( x = a \), as in Figure 1 below.

(a) (4 points) In figure 2, the area is cut into two pieces, and one of the angles is labeled. Use trigonometry and geometry to find the area of the circular wedge in Figure 2. Your answer will be in terms of \( a \).

The area of the entire circle is \( \pi \) since it is the unit circle. The whole circle has “angle” \( 2\pi \). The shaded area in figure 2 is a circular sector that has angle only \( \frac{\pi}{2} - \cos^{-1}a \). So that sector has area

\[
\frac{\pi}{2} - \cos^{-1}a \cdot \frac{\pi}{2\pi} = \frac{\pi}{4} - \frac{1}{2} \cos^{-1}a.
\]

(b) (4 points) Use trigonometry and geometry to find the area of the triangle in terms of \( a \).

The area of a triangle is a bit easier. The base of the triangle is \( a \), and using the Pythagorean Theorem the height of the triangle is \( \sqrt{1 - a^2} \). So the area is

\[
\frac{1}{2}a\sqrt{1 - a^2}.
\]
(c) (5 points) Using your work above, write an antiderivative for the function \( y = \sqrt{1-x^2} \).

This was the big point of the question – an antiderivative tells us the area under the curve. So all we have to do to get the points here is add our answers to (a) and (b), and write everything in terms of \( x \) instead of \( a \). So the answer here is

\[
\frac{\pi}{4} - \frac{1}{2} \cos^{-1} x + \frac{1}{2} x \sqrt{1-x^2}.
\]

(d) (2 points) Check your answer by taking the derivative and simplifying.

The cancellation here is quite neat. The derivative of \( \frac{\pi}{4} \) is 0, and we need product rule to find the derivative of the third term, and the rest is algebra:

\[
\frac{1}{2} \sqrt{1-x^2} + \frac{1}{2} \sqrt{1-x^2} + \frac{1}{2} x \cdot \frac{-2x}{2\sqrt{1-x^2}} = \frac{1}{2} \left( \frac{1}{\sqrt{1-x^2}} + \sqrt{1-x^2} + x \frac{-x}{\sqrt{1-x^2}} \right) = \frac{1}{2} \left( \frac{1}{\sqrt{1-x^2}} + \frac{1-x^2}{\sqrt{1-x^2}} + \frac{-x^2}{\sqrt{1-x^2}} \right) = \frac{1}{2} \left( 2 - 2x^2 \right) = \frac{1-x^2}{\sqrt{1-x^2}} = \sqrt{1-x^2}.
\]

4. The velocity of a particular ball thrown up into the air was given by \( v(t) = 10 - 10t \) meters per second.

(a) (4 points) What was the ball’s net change in height from \( t = 0 \) to \( t = 2 \)?

The net change is simply given by the integral:

\[
\int_0^2 10 - 10t \, dt = 10t - 5t^2 \bigg|_0^2 = 20 - 20 = 0.
\]

(b) (4 points) What was the ball’s total distance traveled from \( t = 0 \) to \( t = 2 \)?

The total distance traveled counts movement up and movement down, so we don’t simply do the integral. Instead we find that the ball was going up from \( t = 0 \) to \( t = 1 \), and down from \( t = 1 \) to \( t = 2 \). Then we do:

\[
\int_0^1 10 - 10t \, dt - \int_1^2 10 - 10t \, dt = 5 + 5 = 10 \text{ meters}.
\]

(c) (6 points) We are also given that the ball’s starting location is \( h(0) = 50 \). Find a formula for the height function \( h(t) \).
The height function is an antiderivative of the velocity function, so we know it has the form $10t - 5t^2 + C$ for some $C$. The starting location simply tells us that $C = 50$, so we can write $h(t) = 10t - 5t^2 + 50$.

5. (10 points) The expression $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{3}{n} \left( 7 + \sin \left( \frac{9i^2}{n^2} \right) \right)$ represents a Riemann sum for a certain area. This question concerns the claim that a Riemann sum is the sum of the areas of lots of small rectangles.

(a) Which part of this expression says “sum”? $\sum_{i=1}^{n}$

(b) Which part of this expression ensures that there are lots of rectangles? $\lim_{n \to \infty}$

(c) Which part of this expression represents the height of the rectangles? $\left( 7 + \sin \left( \frac{9i^2}{n^2} \right) \right)$

(d) Which part of this expression represents the width of the rectangles? $\frac{3}{n}$

(e) Does this expression use left or right endpoints to approximate the area? Right endpoints

(f) Convert this sum into an integral of a function. Do not evaluate the integral.
   There are several possible answers here, but the most common correct answers are:
   $$\int_{0}^{3} 7 + \sin x^2 \, dx \quad \text{and} \quad \int_{0}^{1} 7 + \sin 9x^2 \, dx$$

6. (8 points) Let $K$ be the region between the line $y = 2 + x$ and the parabola $y = x^2$. When we revolve the region $K$ around the $x$-axis, what is the volume of the resulting solid? Simplify to make sure you get a positive number.
   This time the disk/washer method is easiest. The outer radius is $2 + x$ and the inner radius is $x^2$.
   For bounds, we find the intersection points to be where $2 + x = x^2$, namely $x = 2$ and $x = -1$.

   $$\int_{-1}^{2} \pi \left( (2 + x)^2 - (x^2)^2 \right) \, dx = \int_{-1}^{2} \pi \left( 4 + 4x + x^2 - x^4 \right) \, dx$$
   $$= \pi \left( 4x + 2x^2 + \frac{1}{3}x^3 - \frac{1}{5}x^5 \right) \bigg|_{-1}^{2}$$
   $$= \pi \left( 8 + 8 + \frac{8}{3} - \frac{32}{5} \right) - \left( -4 + 2 - \frac{1}{3} + \frac{1}{5} \right)$$
   $$= \frac{72\pi}{5}.$$
7. (8 points) When we revolve the region \( K \) around the vertical line \( x = 2 \), what is the volume of the resulting solid? Simplify to make sure you get a positive number.

The easiest thing to do here is use the shell method. The radius of the shells will be \( 2 - x \), and the height will be \( 2 + x - x^2 \) since the line is above the parabola.

\[
\begin{align*}
\int_{-1}^{2} 2\pi (2-x)(2+x-x^2) \, dx &= \int_{-1}^{2} 2\pi (4 + 2x - 2x^2 - 2x - x^2 + x^3) \, dx \\
&= \int_{-1}^{2} 2\pi (4 - 3x^2 + x^3) \, dx \\
&= 2\pi \left( 4x - x^3 + \frac{1}{4}x^4 \right) \bigg|_{-1}^{2} \\
&= 2\pi \left( 8 - 8 + 4 \right) - \left( -4 + 1 + \frac{1}{4} \right) \\
&= \frac{27\pi}{2}.
\end{align*}
\]

8. (10 points) A middle-school student announces to you that they have invented a new two-dimensional shape, called a “bowtie.” They say that you build this shape by drawing two parabolas that are mirror-images of each other, and shading in the region between.

![Diagram of a bowtie shape]

Using any method, find the area of the “bowtie” of width 4 and height 2.

The easiest way to do this is to draw some axes on the problem so that we can find the equations of the two parabolas. If we do this so that the center point is at the origin, then we have two parabolas of the form \( y = ax^2 \) for some value of \( a \).

Since the width is 4 and height is 2, one of the parabolas contains the point \( (2,1) \) so \( a = 1/4 \), and the other one contains the point \( (2,-1) \) so \( a = -1/4 \). Then we know we are trying to find the area between the two parabolas \( y = \frac{1}{4}x^2 \) and \( y = -\frac{1}{4}x^2 \) between -2 and 2:
\[
\int_{-2}^{2} \frac{1}{4} x^2 - \frac{1}{4} x^2 \, dx = \int_{-2}^{2} \frac{1}{2} x^2 \, dx
\]
\[
= \frac{1}{6} x^3 \bigg|_{-2}^{2}
\]
\[
= \frac{8}{3}
\]

9. (10 points) Using any method, find the volume of the solid whose base is the triangular region bounded by \( y = -2x \), \( y = 2x \), and \( x = 3 \), and whose cross-sections parallel to the y-axis are semicircles with the diameter on the base.

One method is to notice that this shape is half of a cone with radius 6, height 3. So the volume is \( \frac{1}{2} \cdot \frac{1}{3} \cdot \pi \cdot 6^2 \cdot 3 = 18\pi \).

Or we can use the method of cross-sectional areas. The cross-sectional slice at horizontal position \( x \) is a semicircle with radius \( 2x \). This means it has cross-sectional area \( \frac{1}{2} \pi (2x)^2 \). So

\[
V = \int_{0}^{3} \frac{1}{2} \pi (2x)^2 \, dx
\]
\[
= \int_{0}^{3} 2\pi x^2 \, dx
\]
\[
= \frac{2}{3} \pi x^3 \bigg|_{0}^{3}
\]
\[
= 18\pi.
\]