1. Compute the values below.

(a) (4 points) \[ \int_{-2}^{-1} \frac{1}{x} \, dx \]
\[ \ln |x| \bigg|_{-2}^{-1} = \ln 1 - \ln 2 = -\ln 2. \]

(b) (4 points) \[ \int \sec^2 x + \cos x + e^x + \sqrt{x} \, dx \]
Each of these pieces has an antiderivative that we can memorize. Since it is an indefinite integral we need to write \(+C\) at the end.
\[ \tan x + \sin x + e^x + \frac{2}{3}x^{3/2} + C \]

(c) (4 points) \[ \frac{d}{dx} \int_2^x e^{x^2 + 7} \, dt \]
This is an application of the Fundamental Theorem of Calculus Part I.
\[ e^{(2x)^2 + 7} \cdot 2 - e^{x^2 + 7}. \]

2. Compute the integrals below.

(a) (4 points) \[ \int x(x - 2)^{100} \, dx \]
Whenever the problem with an integral is a function inside another one, we should use substitution. Here \(x - 2\) is inside another function, so we use \(u = x - 2, du = dx\).
\[ \int (u + 2)u^{100} \, du = \int u^{101} + 2u^{100} \, du \]
\[ = \frac{1}{102}u^{102} + \frac{2}{101}u^{101} + C \]
\[ = \frac{1}{102}(x - 2)^{102} + \frac{2}{101}(x - 2)^{101} + C. \]

(b) (4 points) \[ \int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx \]
Again the problem is a function inside another one, so we use substitution: Here \(u = \sqrt{x}\) works. Then \(du = \frac{1}{2\sqrt{x}} \, dx\). Our bounds are now from \(u = 2\) to \(u = 3\).
\[ \int_2^3 2e^u \, du = 2e^3 - 2e^2. \]

(c) (4 points) \[ \int_1^4 \frac{x + x^2 + \sqrt{x}}{\sqrt{x}} \, dx \]
There are two ways to do this; you could try a substitution \(u = \sqrt{x}\). I prefer the method of simplifying first, then integrating:
\[
\int_1^4 x + x^2 + \sqrt{x} \, dx = \int_1^4 \sqrt{x} + x^{3/2} + 1 \, dx \\
= \left[ \frac{2}{3} x^{3/2} + \frac{2}{5} x^{5/2} + x \right]_1^4 \\
= \frac{2}{3} 4^{3/2} + \frac{2}{5} 4^{5/2} + 4 - \left( \frac{2}{3} 1^{3/2} - \frac{2}{5} 1^{5/2} - 1 \right) \\
= \frac{16}{3} + \frac{64}{5} + 4 - \frac{2}{3} - \frac{2}{5} - 1 \\
= \frac{301}{15}.
\]

3. Consider the function \( f(x) = \sqrt{1 - x^2} \) between \( x = 0 \) and \( x = a \), as in Figure 1 below.

(a) (4 points) In figure 2, the area is cut into two pieces, and one of the angles is labeled. Use trigonometry and geometry to find the area of the circular wedge in Figure 2. Your answer will be in terms of \( a \).

The area of the entire circle is \( \pi \) since it is the unit circle. The whole circle has “angle” \( 2\pi \). The shaded area in figure 2 is a circular sector that has angle only \( \frac{\pi}{2} - \cos^{-1} a \). So that sector has area

\[
\frac{\frac{\pi}{2} - \cos^{-1} a}{2\pi} \cdot \pi = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} a.
\]

(b) (4 points) Use trigonometry and geometry to find the area of the triangle in terms of \( a \).

The area of a triangle is a bit easier. The base of the triangle is \( a \), and using the Pythagorean Theorem the height of the triangle is \( \sqrt{1 - a^2} \). So the area is

\[
\frac{1}{2} a \sqrt{1 - a^2}.
\]

(c) (5 points) Using your work above, write an antiderivative for the function \( y = \sqrt{1 - x^2} \).

This was the big point of the question – an antiderivative tells us the area under the curve. So all we have to do to get the points here is add our answers to (a) and (b), and write everything in terms of \( x \) instead of \( a \). So the answer here is
\[
\frac{\pi}{4} - \frac{1}{2} \cos^{-1} x + \frac{1}{2} x \sqrt{1 - x^2}.
\]

(d) (2 points) Check your answer by taking the derivative and simplifying. The cancellation here is quite neat. The derivative of \(\frac{\pi}{4}\) is 0, and we need product rule to find the derivative of the third term, and the rest is algebra:

\[
\frac{1}{2} \frac{1}{\sqrt{1 - x^2}} + \frac{1}{2} \sqrt{1 - x^2} + \frac{1}{2} x \frac{-2x}{2\sqrt{1 - x^2}} \\
= \frac{1}{2} \left( \frac{1}{\sqrt{1 - x^2}} + \sqrt{1 - x^2} + x \frac{-x}{\sqrt{1 - x^2}} \right) \\
= \frac{1}{2} \left( \frac{1}{\sqrt{1 - x^2}} + \frac{1 - x^2}{\sqrt{1 - x^2}} + \frac{-x^2}{\sqrt{1 - x^2}} \right) \\
= \frac{1}{2} \frac{2 - 2x^2}{\sqrt{1 - x^2}} \\
= \frac{1 - x^2}{\sqrt{1 - x^2}} \\
= \sqrt{1 - x^2}.
\]

4. The velocity of a particle moving along the x-axis is given by \(v(t) = t + \sin t\).

(a) (4 points) What was the particle’s net change in distance from \(t = -\pi\) to \(t = \pi\)?
Net change in distance is given by the integral of the velocity. So we find

\[
\int_{-\pi}^{\pi} t + \sin t \, dt = \left. \frac{1}{2} t^2 - \cos t \right|_{-\pi}^{\pi} \\
= \frac{1}{2} \pi^2 - \cos \pi - \frac{1}{2} \pi^2 + \cos -\pi \\
= 0.
\]
You could skip the integral by noting that this is the integral of an odd function from \(-1\) to 1, so it must be zero.

(b) (4 points) What was the particle’s total distance traveled from \(t = -\pi\) to \(t = \pi\)?
The total distance traveled counts movement to the right and movement to the left both positive, so we don’t simply do the integral. Instead we find that the particle moved to the left from \(t = -\pi\) to \(t = 0\), and to the right from \(t = 0\) to \(t = \pi\). Then we do:
$D = -\int_{-\pi}^{0} t + \sin t \, dt + \int_{0}^{\pi} t + \sin t \, dt$

$= -\left(\frac{1}{2}t^2 - \cos t\right) \bigg|_{-\pi}^{0} + \left(\frac{1}{2}t^2 - \cos t\right) \bigg|_{0}^{\pi}$

$= -\left(0 - \cos 0 - \frac{1}{2}\pi^2 + \cos -\pi\right) + \left(\frac{1}{2}\pi^2 - \cos \pi - 0 + \cos 0\right)$

$= 4 + \pi^2$.

(c) (6 points) We are also given that the particle’s starting location is $x(0) = 10$. Find a formula
for the position $x(t)$.

The height function is an antiderivative of the velocity function, so we know it has the form
$\frac{1}{2}t^2 - \cos t + C$ for some $C$. The starting location simply tells us that $0 - \cos 0 + C = 10$, so we
can solve to find that $C = 11$ and $h(t) = \frac{1}{2}t^2 - \cos t + 11$.

5. (4 points) Approximate the area under the curve $y = (1 + 2x + x^3)^{3/2}$ from $x = 0$ to $x = 2$ by using
left endpoints and two rectangles. Simplify your answer.

The width of the rectangles is 1, and the heights are $f(0)$ and $f(1)$ where $y = f(x)$. So we need to
find $f(0)$ and $f(1)$, then compute $1 \cdot f(0) + 1 \cdot f(1)$.

$f(0) = (1 + 0 + 0)^{3/2} = 1$.  $f(1) = (1 + 2 + 1)^{3/2} = 4^{3/2} = 8$.

So the estimated area is $1 \cdot 1 + 1 \cdot 8 = 9$.

6. (2 points) Is this an underestimate or an overestimate for the actual area? Why?

This is an underestimate for the actual area. Left endpoints are an underestimate for increasing
functions, and this is an increasing function.

7. (4 points) Draw a picture that shows a rough sketch of the function. Shade in the area that you
found in part (a). This is different from the area under the curve.

8. (8 points) Let $K$ be the region between the x-axis ($y = 0$) and the function $y = 1 - x^2$. When we
revolve the region $K$ around the x-axis, what is the volume of the resulting solid? Simplify to make
sure you get a positive number.
First we find that the intersection points of the x-axis and \( y = 1 - x^2 \) are at \((-1, 0)\) and at \((1, 0)\).

Now our functions are already solved for \( y \), so we will probably try to do integrals with respect to \( x \) whenever possible. In this case, doing our integral with respect to \( x \) leads us to using the disk method.

The radius of the disk is \( 1 - x^2 \), so our integral is:

\[
\int_{-1}^{1} \pi(1 - x^2)^2 \, dx
\]

\[
= \int_{-1}^{1} \pi(1 - 2x^2 + x^4) \, dx
\]

\[
= \pi \left( x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right) \bigg|_{-1}^{1}
\]

\[
= \pi \left( 1 - \frac{2}{3} + \frac{1}{5} \right) - \pi \left( -1 + \frac{2}{3} - \frac{1}{5} \right)
\]

\[
= 16\pi/15.
\]

9. (8 points) When we revolve the region \( K \) around the vertical line \( x = 2 \), what is the volume of the resulting solid? Simplify to make sure you get a positive number.

Now we will use the shell method. The radius of the shell is \( 2 - x \), and the height is \( 1 - x^2 \). The volume is

\[
V = \int_{-1}^{1} 2\pi(2 - x)(1 - x^2) \, dx
\]

\[
= \int_{-1}^{1} 2\pi(2 - x - 2x^2 + x^3) \, dx
\]

\[
= 2\pi \left( 2x - \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 \right) \bigg|_{-1}^{1}
\]

\[
= 2\pi \left( 2 - \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) - 2\pi \left( -2 - \frac{1}{2} + \frac{2}{3} + \frac{1}{4} \right)
\]

\[
= 16\pi/3.
\]

10. (10 points) A middle-school student announces to you that they have invented a new two-dimensional shape, called a “football.” They say that you build this shape by drawing two parabolas that are mirror images of each other, and shading in the region between them.
Using any method, find the area of the “football” of width 4 and height 2.

We need to find the area between two parabolas. The best way to do this is to find the equations of the parabolas so that we can do an integral to find the area.

We can start by putting the axes so that the origin (0,0) is right in the middle of the football. This means that:

The top parabola goes through the points (-2, 0), (0, 1), and (2, 0).

So we can tell that the equation of the top parabola is $y = 1 - ax^2$.

Using one of the other two points, we find $a = 1/4$, so the top parabola is $y = 1 - \frac{1}{4}x^2$ and the bottom parabola is $y = \frac{1}{4}x^2 - 1$.

Now the area between the two curves is the integral:

$$
\int_{-2}^{2} \left(1 - \frac{1}{4}x^2\right) - \left(\frac{1}{4}x^2 - 1\right) \, dx = \int_{-2}^{2} \left(2 - \frac{1}{2}x^2\right) \, dx
$$

$$
= 2x - \frac{1}{6}x^3 \bigg|_{-2}^{2}
$$

$$
= 4 - \frac{8}{6} + 4 - \frac{8}{6}
$$

$$
= \frac{16}{3}.
$$

11. (10 points) Using any method, find the volume of the solid whose base is the triangular region bounded by $y = 0$, $y = 2x$, and $x = 3$, and whose cross-sections parallel to the y-axis are squares.

One method is to notice that this shape is a square pyramid with base side length 6 and height 3. So the volume is $\frac{1}{3} \cdot 6^2 \cdot 3 = 36$.

Or we can use the method of cross-sectional areas. The cross-sectional slice at horizontal position $x$ is a square with side length $2x$. This means it has cross-sectional area $(2x)^2$. So
\[ V = \int_0^3 (2x)^2 \, dx \]
\[ = \frac{4}{3} x^3 \bigg|_0^3 \]
\[ = 36. \]