1. Compute the following limits. If you use L’Hospital’s rule, include the indeterminate form that you found and indicate the step where you used L’Hospital’s rule.

(a) (4 points) \( \lim_{x \to 0} \frac{7x}{\sin(3x)} \)
This gives the indeterminate form \( \frac{0}{0} \), so we can use L’Hospital’s rule directly to get \( \lim_{x \to 0} \frac{7}{3\cos(3x)} = \frac{7}{3} \).

(b) (4 points) \( \lim_{x \to 0} \frac{x}{\sqrt{x^2 + 1}} \)
This function is continuous at \( x = 0 \) so we can find the value of the limit to be \( \frac{0}{1} = 0 \). Using L’Hospital’s rule here will give an incorrect answer.

(c) (6 points) \( \lim_{x \to 1} x^{1/(x^2-1)} \)
This gives the indeterminate form \( 1^\infty \), so we need to do some work before using L’Hospital’s rule. We can add an exponential and a logarithm to get:

\[
e^{\lim_{x \to 1} \ln(x^{1/(x^2-1)})} = e^{\lim_{x \to 1} \frac{1}{x^2-1} \ln x}
= e^{\lim_{x \to 1} \frac{\ln x}{x^2-1}}
\]

From here, we see the indeterminate form \( \frac{0}{0} \), so we can use L’Hospital’s rule to find

\[
e^{\lim_{x \to 1} \frac{\ln x}{x^2-1}} = e^{\lim_{x \to 1} \frac{1/x}{2x}}
= e^{\lim_{x \to 1} \frac{1}{x^2}}
= e^{\frac{1}{2}}.
\]

2. Let \( f(x) = 2x + \sin x \) on the interval \([0, \pi]\).

(a) (6 points) Without actually finding the absolute maximum value, we know that this function has an absolute maximum value. How?
This is a continuous function on a closed interval. As a result, the Extreme Value Theorem tells us that it has an absolute maximum somewhere.

(b) (4 points) Does \( f(x) \) have any critical points on the interval \([0, \pi]\)? (Show work for all parts!)
To find critical points we need to look for places where the derivative is zero, and places where the derivative is undefined. The derivative is \( f'(x) = 2 + \cos x \), which is defined everywhere. So we next check for places where it equals zero. But \( \cos x \) is never equal to -2, so the derivative is never zero either. This means there are no critical points.
(c) (6 points) Find the absolute maximum and minimum values of $f(x)$ on the interval $[0, \pi]$.

Since there are no critical points and the function is continuous, we only need to test the endpoints to see which is the maximum and which is the minimum. We get $f(0) = 0$ and $f(2\pi) = 2\pi$, so the absolute maximum value is $2\pi$ and the absolute minimum value is 0.

3. Let $f(x) = \sqrt{x}$.

(a) (3 points) Find the linearization of $f(x)$ at $x = 100$.

$L(x) = f(100) + f'(100)(x - 100) = 10 + \frac{1}{20}(x - 100)$.

(b) (3 points) Explain why this linearization is not very useful for estimating the value of $\sqrt{5}$.

This linearization is only a good approximation for $f(x)$ near $x = 100$. But 5 is very far away from 100, so $f(5)$ and $L(5)$ might be very far apart!

(c) (6 points) Find an estimate for $\sqrt{5}$ using a different linearization of $f(x) = \sqrt{x}$. Simplify! Your answer should not include any square roots in it.

We can find a different linearization by choosing a different number to plug in instead of 100. The easiest one to choose is $a = 4$. So we find

$L(x) = f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4)$.

Our estimate for $\sqrt{5}$ is $L(5) = 2 + \frac{1}{4}(5 - 4) = 2.25$.

4. (10 points) A population grows by the rule

$$\frac{dP}{dt} = \frac{1}{100} P \text{ members/year}$$

How long does it take for this population to double?

The big idea from the exponential growth section was that we know the population follows the formula

$$P(t) = P_0 e^{\frac{1}{100} t}$$

We are trying to find the time where $P(t) = 2P_0$, so we set the population equal to $2P_0$ and solve for $t$.

$$2P_0 = P_0 e^{\frac{1}{100} t}$$
$$2 = e^{\frac{1}{100} t}$$
$$\ln 2 = \frac{1}{100} t$$
$$100 \ln 2 = t$$
5. (8 points) A particle $P$ is moving along the curve $y = 1/x$ such that its horizontal speed $\frac{dx}{dt}$ is always exactly 3. What is the vertical speed of the particle when it reaches the point $(2, \frac{1}{2})$?

We have $y = \frac{1}{x}$, so we can use implicit differentiation to find a relation between $\frac{dy}{dt}$ and $\frac{dx}{dt}$:

$$\frac{dy}{dt} = \frac{1}{x^2} \frac{dx}{dt}$$

We know the particle’s horizontal speed and its location, so we plug in those quantities:

$$\frac{dy}{dt} = -\frac{1}{2^2} \cdot 3 = -\frac{3}{4}$$

6. (8 points) A spinning light source from a siren is situated 20 feet from a point $Q$ on a straight wall. The light source spins 2.5 times per second and emits a beam of light. How fast is the beam of light moving along the wall when it passes the point $Q$?

We can draw a right triangle that illustrates the situation.

![Diagram](image)

From this we can write the relation $\tan \theta = \frac{y}{20}$.

Using implicit differentiation we find that

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{20} \frac{dy}{dt}$$

Since $\frac{d\theta}{dt}$ is 2.5 revolutions per second, we convert that:

$$2.5 \text{ revolutions/second} \cdot \frac{2\pi}{\text{revolution}} = \frac{5\pi}{\text{second}}$$

Then $\frac{dy}{dt} = 20 \sec^2 \theta \cdot 5\pi = 100\pi \frac{ft}{s}$.

7. Suppose you have a function $f(x)$ with $f(3) = 7$ and $f(5) = 13$.

(a) (6 points) If $f(x)$ is differentiable, what exactly does the mean value theorem tell you about the derivative of $f$?

The mean value theorem works here because $f(x)$ is differentiable. It tells us that there is some value $c$ in the interval $(3, 5)$ where $f'(c)$ is the same as the average rate of change on the interval. The average rate of change is $\frac{13-7}{5-3} = 6$, so we have some $c$ in the interval $(3, 5)$ such that $f'(c) = 6$. 

(b) (4 points) Sketch the graph of a function \( f(x) \) that is continuous but not differentiable, has \( f(3) = 7 \), and \( f(5) = 13 \), but so that your answer to part (a) is false for this function. Many possible answers, all of them have some kind of cusp or corner though. Be careful; a cusp or corner does not automatically make the mean value theorem fail, though!

8. The function \( g(x) = e^{7x} + e^{-x} \) has one local extreme value, no asymptotes, and is always concave up.

(a) (8 points) Find the local extreme value, and use the first derivative test to see whether it is a local maximum or minimum value.

We will find the local extreme value by looking for critical points. We start by taking the derivative. Since the derivative exists everywhere, we only have to set it equal to zero.

\[
g'(x) = 7e^{7x} - e^{-x} \\
0 = 7e^{7x} - e^{-x} \\
e^{-x} = 7e^{7x} \\
\ln e^{-x} = \ln(7e^{7x}) \\
-x = \ln(7) + 7x \\
-\ln 7 = 8x \\
\frac{-\ln 7}{8} = x
\]

We then use the first derivative test by testing \( x \)-values on either side of this one. \( \ln 7 \) is approximately 2, so we will use \( x = -1 \) on one side and \( x = 0 \) on the other side.

\( g'(-1) = 7e^{-7} - e^{-1} \) which is definitely negative.

\( g'(0) = 7 - 1 = 6 \) which is positive.

So the critical point at \( x = \frac{-\ln 7}{8} \) is a local minimum.
(b) (4 points) Sketch a graph of the function.

9. (10 points) A company knows that if it sets the price of a widget at $x$, it can sell $2000 - x$ widgets. Find the maximum revenue for this company.

The revenue function here is given by $R(x) = x(2000 - x)$. So we can find the maximum value by first finding endpoints and then critical points of this function.

The endpoints for this function are $x = 0$ and $x = 2000$. The critical point is found by taking the derivative:

$$R'(x) = 2000 - 2x$$

This equals zero at $x = 1000$.

So the only possible maximum values are $R(0) = 0$, $R(1000) = 1000000$, or $R(2000) = 0$. The largest one is 1000000, so this is the maximum revenue.