Math 221, EL1 - Test #1 - September 27, 2012

1. Let \( f(x) = \begin{cases} x^2 + 2 & x < 1 \\ x^2 + 6x + 9 & x \geq 1 \end{cases} \)

   (a) (2 points) Find \( \lim_{x \to 0} f(x) \). No work needs to be shown.
   For \( x \) near 0, \( f(x) = x^2 + 2 \). So the answer is \( 0^2 + 2 = 2 \).

   (b) (4 points) Find \( \lim_{x \to 2} f(x) \) or explain why it does not exist.
   For \( x \) near 2, \( f(x) = x^2 + 6x + 9 \). So the answer is \( 2^2 + 6 \times 2 + 9 = 25 \).

   (c) (4 points) Find \( \lim_{x \to 1} f(x) \) or explain why it does not exist.
   Comparing the two sides of the limit, \( \lim_{x \to 1^-} f(x) = 1 + 2 = 3 \) and \( \lim_{x \to 1^+} f(x) = 1 + 6 + 9 = 16 \).
   Since the two do not agree, the limit does not exist.

2. (10 points) Use the definition of the derivative to find the derivative of \( y = x^2 + x \).

\[
\frac{dy}{dx} = \lim_{h \to 0} \frac{(x + h)^2 + (x + h) - x^2 - x}{h}
= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h}
= \lim_{h \to 0} \frac{2xh + h^2 + h}{h}
= \lim_{h \to 0} 2x + h + 1
= 2x + 1
\]

3. Let \( g(x) = |x^2 - 4| \).

   (a) (8 points) Write a limit expression for \( g'(2) \) by using the definition of the derivative. Simplify your answer so that it does not contain any \( g \)'s or \( x \)'s.

\[
g'(2) = \lim_{h \to 0} \frac{g(2 + h) - g(2)}{h}
= \lim_{h \to 0} \frac{|(2 + h)^2 - 4| - |2^2 - 4|}{h}
= \lim_{h \to 0} \frac{|4 + 4h + h^2 - 4| - |2^2 - 4|}{h}
= \lim_{h \to 0} \frac{|4h + h^2|}{h}
\]

   (b) (3 points) Your answer to (a) should include a limit as \( h \) approaches 0. What do you get if you plug in \( h = -0.1 \)? Simplify your answer completely.

\[
\left| \frac{-0.4 + 0.01}{-0.1} \right| = -3.9
\]
(c) (3 points) Your answer to (a) should include a limit as \( h \) approaches 0. What do you get if you plug in \( h = 0.1 \)? Simplify your answer completely.

\[
\frac{|0.4 + 0.01|}{0.1} = 4.1
\]

(d) (6 points) Explain why \( g'(2) \) does or does not exist. Bring the answers to the previous two questions into your argument. (2-3 sentences)

\( g'(2) \) is represented by a limit as \( h \) approaches 0. Since the limits from left and right do not agree (see (b) and (c)), the limit does not exist. So \( g'(2) \) does not exist.

4. (3 points per part) Compute \( \frac{dy}{dx} \) in the following cases using any method we have studied so far. Your answer may include both \( x \)'s and \( y \)'s. You do not need to show work for some of these.

(a) \( y = (3x + 5)^2 \)
\[
\frac{dy}{dx} = 2(3x + 5)(3)
\]

(b) \( y = e^{3x^2} + \pi^x + 3^x \)
\[
\frac{dy}{dx} = 0
\]

(c) \( y = \tan^{-1}(x^4) \)
\[
\frac{dy}{dx} = \frac{1}{1 + x^8}(4x^3)
\]

(d) \( y^3 + y = x^2 + x \)
Implicit differentiation:

\[
3y^2 \frac{dy}{dx} + \frac{dy}{dx} = 2x + 1
\]
\[
(3y^2 + 1) \frac{dy}{dx} = 2x + 1
\]
\[
\frac{dy}{dx} = \frac{2x + 1}{3y^2 + 1}
\]

5. Compute the following limits. Do not use L'Hopital’s Rule, which we have not yet studied. Show at least one step of work for every limit.

(a) (3 points) \( \lim_{x \to 3} \frac{x^2 - 2x - 3}{x+3} \)

We can plug in \( x = 3 \) here since the function is continuous at 3, so \( \frac{9 - 6 - 3}{6} = 0 \).

(b) (3 points) \( \lim_{x \to 3} \frac{x^2 - 2x - 3}{x-3} \)

When we try to plug in \( x = 3 \) we get \( \frac{0}{0} \), so we factor and cancel:

\[
\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 1)}{x - 3}
\]
\[
= \lim_{x \to 3} x + 1
\]
\[
= 4
\]
(c) (6 points) \( \lim_{x \to \infty} \sqrt{x^2 + 7x} - \sqrt{x^2 + 4x} \)

\[
\lim_{x \to \infty} \sqrt{x^2 + 7x} - \sqrt{x^2 + 4x} = \lim_{x \to \infty} \frac{(\sqrt{x^2 + 7x} - \sqrt{x^2 + 4x})(\sqrt{x^2 + 7x} + \sqrt{x^2 + 4x})}{\sqrt{x^2 + 7x} + \sqrt{x^2 + 4x}} \\
= \lim_{x \to \infty} \frac{x^2 + 7x - x^2 - 4x}{\sqrt{x^2 + 7x} + \sqrt{x^2 + 4x}} \\
= \lim_{x \to \infty} \frac{3x}{\sqrt{x^2 + 7x} + \sqrt{x^2 + 4x}} \\
= \lim_{x \to \infty} \frac{3}{x} \left( \frac{\sqrt{x^2 + 7x} + \sqrt{x^2 + 4x}}{\sqrt{1 + 7/x} + \sqrt{1 + 4/x}} \right) \\
= \lim_{x \to \infty} \frac{3}{\sqrt{1 + 7/x} + \sqrt{1 + 4/x}} \\
= \frac{3}{2}
\]

6. (5 points) The table below gives estimates for world population \( P(t) \) in year \( t \). \( P(t) \) is given in millions of people.

<table>
<thead>
<tr>
<th>( t )</th>
<th>1750</th>
<th>1800</th>
<th>1850</th>
<th>1900</th>
<th>1950</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(t) )</td>
<td>791</td>
<td>978</td>
<td>1262</td>
<td>1650</td>
<td>2519</td>
<td>6070</td>
</tr>
</tbody>
</table>

Try to use the Intermediate Value Theorem to show that at some point, there were exactly 900 million people on Earth. What goes wrong?

The function \( P(t) \) is not continuous since it always jumps by at least one at a time, so the Intermediate Value Theorem cannot be applied.

7. (10 points) Show that there is some value of \( x \) such that \( x^4 + x = 1 \). Hint: You do not need to actually find the value of \( x \).

The function \( x^4 + x \) is continuous everywhere since it is a polynomial. So we can use the Intermediate Value Theorem.

At \( x = 0 \), \( x^4 + x = 0 \). At \( x = 2 \), \( x^4 + x = 18 \). Since 1 is between 0 and 18, there is definitely some point \( c \) between 0 and 2 such that \( c^4 + c = 1 \).

8. Below is a graph of height \( h(t) \), given in meters, between \( t = -3 \) and \( t = 9 \). \( t \) is given in hours.

(a) (5 points) Sketch a graph of the velocity \( h'(t) \). For the purposes of this problem, do not worry about whether the derivative of the function is defined or not at the endpoints of the graph.
(b) (3 points) What is the average velocity from $t = 0$ to $t = 3$? Include the units.

0 meters/hour

(c) (3 points) What is the instantaneous velocity at $t = -2$? Include the units.

It is the same as the slope of that line segment, so we calculate the slope and get -1 meter/hour.
9. A falling object’s height at time $t$ is $f(t) = 100 - 10t^2$ meters where $t$ is in seconds.

(a) (1 point) Sketch a graph of $f(t)$.

(b) (1 point) Add a secant line to the graph connecting the points at $t = 1$ and $t = 1 + h$ where $h = 1.5$. Label it “(b)”.

(c) (2 points) What physical quantity does the slope of that line represent? Do not simply calculate the slope.
   The average velocity from $t = 1$ to $t = 2.5$.

(d) (1 point) Draw a tangent line to the graph at $t = 1$. Label it “(d)”.

(e) (2 points) What physical quantity does the slope of that line represent? Do not simply calculate the slope.
   The instantaneous velocity at $t = 1$.

(f) (1 point) Above you drew a secant line with $h = 1.5$. What value of $h$ would cause a secant line that was closer to the tangent line?
   Any value of $h$ that is closer to zero but not 0. For example $h = 1$ would work.

(g) (2 points) What happens as $h$ gets very close to 0? (1-2 sentences)
   The secant line gets closer and closer to being a tangent line. This is because the average velocity gets closer and closer to being the instantaneous velocity.