Math 221, BL1 - Test #3 - December 1, 2011

Name:____________________________________________________________

Signature:________________________________________________________

Circle your discussion section:  BD1 (8:00, Michelle)  BD2 (9:00, Michelle)
   BD3 (10:00, Sepideh)  BD4 (12:00, Khang)  BD5 (11:00, Khang)
   BD6 (1:00, Juan)  BD7 (2:00, Juan)  BD8 (3:00, Sepideh)

DO NOT OPEN EXAM UNTIL TOLD TO DO SO. SIT IN THE SEAT CIRCLED BELOW.

Time: 50 minutes. You may not use any books or calculator; you may use a single handwritten notecard. There are 100 points possible. To get any credit, you must show your work. Unless indicated, you do not need to simplify your answers. Partial credit will be based only on what is actually written on the paper. All intermediate steps should be correct as written.

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1. (6 points per part) Evaluate the integrals.

(a) \[ \int_\pi^{2\pi} \sin x \, dx \]

The antiderivative of \( \sin x \) is \( -\cos x \), so we evaluate \( -\cos x \) at \( \pi \) and \( 2\pi \).

\[ -\cos 2\pi - -\cos \pi = -1 - -(-1) = -2. \]

(b) (6 points) \[ \int_1^2 xe^x^2 \, dx \]

Several substitutions will work here, including \( u = x^2 \), \( u = e^x^2 \), and \( u = e^{\frac{1}{2}x^2} \). The most common solution is with \( u = x^2 \), \( du = 2x \, dx \):

\[ \int_{1^2}^{2^2} \frac{1}{2} e^u \, du = \frac{1}{2} e^u \bigg|_1^4 = \frac{1}{2} e^4 - \frac{1}{2} e. \]

(c) (6 points) \[ \int \frac{3x^2}{1+e^2} \, dx \]

A substitution with \( u = x^3 \) and \( du = 3x^2 \, dx \) is the right plan here. This gives us:

\[ \int \frac{du}{1+u^2} = \tan^{-1} u + C = \tan^{-1}(x^3) + C. \]

2. (6 points per part) Evaluate.

(a) \[ \int_0^{\pi/4} \left( \frac{d}{dx} e^{\tan x} \right) \, dx \]

We need to find an antiderivative for \( \frac{d}{dx} e^{\tan x} \), then evaluate it at the two limits of integration.

So we get:

\[ e^{\tan x} \bigg|_0^{\pi/4} = e^{\tan \pi/4} - e^{\tan 0} = e - 1. \]

(b) \[ \frac{d}{dx} \left( \int_0^{\pi/4} e^{\tan t} \, dt \right) \]

\( \int_0^{\pi/4} e^{\tan t} \, dt \) is hard to evaluate, but we know it is some number. Thinking in terms of \( x \), this number is a constant, so its derivative is 0.

(c) \[ \frac{d}{dx} \left( \int_{\pi/2}^{\pi/4} e^{\tan t} \, dt \right) \]

We use the fundamental theorem of calculus:

\[ \frac{d}{dx} \left( \int_{\pi/2}^{\pi/4} e^{\tan t} \, dt \right) = e^{\tan \pi/4} \cdot 0 - e^{\tan x} \cdot 1 = -e^{\tan x}. \]
3. (5 points per part) Let \( v(t) \) be the velocity (in meters per second) at time \( t \) seconds of a person moving in a straight line. A graph of \( v(t) \) is given.

Let \( A_1 \) be the area of the first region and \( A_2 \) be the area of the second region. Note that \( A_1 \) and \( A_2 \) are both positive numbers.

(a) Write an expression using integrals that calculates the value \( A_1 \).

For a positive function, the area under its graph is the same as the integral of that function:

\[
\int_0^2 v(t) \, dt.
\]

(b) Write an expression using integrals that calculates the value \( A_1 + A_2 \).

Negative functions don’t work the same way: if you integrate a negative function, you get the opposite of the area. So, to find \( A_1 + A_2 \), we need:

\[
\int_0^2 v(t) \, dt - \int_2^5 v(t) \, dt.
\]

(c) In words (without integrals), what does \( A_1 + A_2 \) represent?

\( A_1 + A_2 \) is the total distance traveled by the person. Whether the person is walking forward or backward, all this distance is counted.

(d) In words (without integrals), what does \( A_1 - A_2 \) represent?

\( A_1 - A_2 \) is the net change in distance, or the displacement of the person. It measures the net effect of all the person’s walking.

4. (6 points per part) The following questions concern the function \( f(x) = \sqrt{4 - x^2} \).

(a) Use any method (but show your work) to evaluate \( \int_0^2 \sqrt{4 - x^2} \, dx \).

This function represents the top half of a circle of radius 2. So, the integral represents the area of a quarter of a circle of radius 2. This area is:

\[
\frac{1}{4} \pi 2^2 = \pi.
\]

(b) Write a limit of Riemann sums that represents \( \int_0^2 \sqrt{4 - x^2} \, dx \).

Dividing the interval into \( n \) subintervals gives a width of \( \frac{2}{n} \). The height of the \( i^{th} \) rectangle will be given by \( f(\frac{2i}{n}) \). So we get

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n} \sqrt{4 - \left(\frac{2i}{n}\right)^2}.
\]
5. Find the total area enclosed by the curves given.

(a) (6 points) \( y = e^x, \ y = -e^x, \ x = 0, \ x = 1 \)

To get the area between two curves \( f(x) \) and \( g(x) \), we must integrate their difference \( |f(x) - g(x)| \). Since the picture shows that \( e^x > -e^x \), we can simply integrate \( e^x - (-e^x) \) instead:

\[
\int_0^1 e^x - (-e^x) \, dx = \int_0^1 2e^x \, dx = 2e^x \bigg|_0^1 = 2e - 2.
\]

(b) (10 points) \( y = x^3 - x, \ y = 3x \)

To get the area between two curves \( f(x) \) and \( g(x) \), we must integrate their difference \( |f(x) - g(x)| \). We need to find out which function is larger by first setting them equal to each other:

\[ x^3 - x = 3x \Rightarrow x^3 - 4x = 0 \Rightarrow x(x^2 - 4) = 0 \Rightarrow x = 0 \text{ or } x = \pm 2. \]

Testing values shows us that \( x^3 - x > 3x \) on the interval \([-2, 0]\) and the opposite is true on \([0, 2]\), so we integrate:

\[
\int_{-2}^0 x^3 - x - 3x \, dx + \int_0^2 3x - x^3 + x \, dx = \int_{-2}^0 x^3 - 4x \, dx + \int_0^2 4x - x^3 \, dx
\]

\[
= \frac{1}{4}x^4 - 2x^2 \bigg|_{-2}^0 + 2x^2 - \frac{1}{4}x^4 \bigg|_0^2
\]

\[
= 0 + 0 - 4 + 8 + 8 - 4 - 0 - 0
\]

\[
= 8.
\]

Notice that the two functions you end up integrating are opposites of one another. This always happens, because of the issues in problem 3; when we are trying to find area, the integral can mix up by counting some area negative if we are not careful.
6. (8 points per part) The region enclosed by \( y = x \) and \( y = x^2 \) is rotated about the line \( x = 2 \).

First we find the bounds of our region by setting the two equations equal and solving:
\[ x = x^2 \text{ when } x^2 - x = 0, \text{ so } x = 0 \text{ or } 1. \]
The bounds for \( y \) are the same since we are using \( y = x \) and \( y = x^2 \).

(a) Find the volume of the resulting solid using the washer method.

We must integrate with respect to \( y \) in order to get washers here. So we need to find the outer radius and the inner radius.

A common mistake was to use \( 2 - x \) and \( 2 - x^2 \), but we must integrate with respect to \( y \), using \( x = y \) and \( x = \sqrt{y} \) instead.

So, we can use \( R = 2 - y \) and \( r = 2 - \sqrt{y} \) in the washer formula:

\[
\pi \int_0^1 (2 - y)^2 - (2 - \sqrt{y})^2 \, dy = \pi \int_0^1 4 - 4y + y^2 - 4 + 4\sqrt{y} - y \, dy
\]
\[
= \pi \int_0^1 -5y + y^2 + 4\sqrt{y} \, dy
\]
\[
= \pi \left( -\frac{5}{2} y + \frac{1}{3} y^2 + \frac{8}{3} y^{3/2} \right)_0^1
\]
\[
= \pi \left( -\frac{5}{2} + \frac{1}{3} + \frac{8}{3} \right)
\]
\[
= \frac{\pi}{2}.
\]

(b) Find the volume of the resulting solid using the shell method.

Using the shell method, we can integrate with respect to \( x \). So we find

\[
2\pi \int_0^1 (2 - x)(x - x^2) \, dx = 2\pi \int_0^1 2x - 3x^2 + x^3 \, dx
\]
\[
= 2\pi \left( x^2 - x^3 + \frac{1}{4} x^4 \right)_0^1
\]
\[
= 2\pi \left( 1 - 1 + \frac{1}{4} \right)
\]
\[
= \frac{\pi}{2}.
\]