Research Statement

Craig Davis

Conformally Flat Spaces of Bounded Curvature

Metric spaces with curvature bounded below (or above) by $K$ in the sense of Alexandrov are generalizations of Riemannian spaces. One starts with a metric space with intrinsic metric, $(X, \rho)$, and then obtains a notion of curvature by comparing the sum of the internal angles of triangles with the internal angles of triangles in a model space. This notion is similar to the sectional curvature of a Riemannian space. Riemannian spaces with sectional curvatures bounded below (or above) by $K$ are spaces with curvature bounded below (or above) by $K$. Examples of non-Riemannian spaces of bounded curvature include two-dimensional polyhedra, trees, convex surfaces, and limits of sequences of Riemannian spaces with bounded curvature subject to certain general restrictions. These spaces are currently of interest and are being studied by Gromov, Cheeger, Schoen and others.

If the set $X$ is a two dimensional manifold, an integral curvature idea can be considered instead. These are the two-dimensional manifolds with bounded curvature and they have been studied extensively by Alexandrov, Zalgaller, Reshetnyak and others. Reshetnyak has provided an analytic characterization of the metric in terms of $\delta$-subharmonic functions (functions that are the difference of two subharmonic functions). Locally the metric of two dimensional Riemannian manifolds can be expressed in terms of an isothermal coordinate system with line element $ds^2 = \lambda(x, y)(dx^2 + dy^2)$, where $\ln \lambda(x, y)$ is a $\delta$-subharmonic function. Reshetnyak shows that this holds for all two dimensional spaces of bounded curvature. From this it can be shown that such spaces can be approximated by Riemannian spaces, and in fact the two dimensional manifolds of bounded curvature form the closure of the set of two dimensional Riemannian spaces. Examples of such spaces include the regular Euclidean cones with curvature $\omega$, which can be represented by $ds^2 = C\left(\frac{1}{x^2+y^2}\right)^{\omega/2\pi} (dx^2 + dy^2)$.

Nikolaev has proved that metric spaces with curvature bounded above by $K$ and below by $K'$ (in the sense of Alexandrov) can be approximated by Riemannian manifolds with sectional curvature bounded above by $K + \epsilon$ and below by $K' - \epsilon$, and in fact there is a coordinate system for which the metric tensor has second Sobolev derivatives. For spaces with one sided curvature bounds approximation results of this form are not possible. Petersen, Wilhelm, and Zhu have given examples of spaces with curvature bounded below by $K$ that cannot be approximated by Riemannian spaces with curvature bounded below by $K - 1/4$.

An important difference between the two dimensional case and the higher dimensional cases appears to be the existence of an isothermal coordinate system, that is, one with line element $ds^2 = \lambda(x)^2dx^2$. So if $X \subset \mathbb{R}^n$, and $\lambda : X \to [0, \infty]$ (which is non-zero on a sufficiently large set and satisfies certain regularity conditions),

$$\rho(x, y) = \inf\left\{ \int_{\Gamma} \lambda(t)|d\Gamma| : \Gamma \subset X \text{ is a simple rectifiable curve from } x \text{ to } y \right\}.$$
A metric space possessing such a coordinate system is said to be conformally flat. One can determine whether or not a Riemannian space is conformally flat by examining its Weyl tensor. Subject to certain regularity conditions on $\lambda$, such spaces generate the same topology as the Euclidean metric on $\mathbb{R}^n$, the angle between rectifiable curves is the same as the Euclidean angle between the curves, and the conformal factor, $\lambda$, can be recovered as a ratio $\rho$ to the Euclidean metric.

The sectional curvature of conformally flat Riemannian spaces is easily calculated. If $ds^2 = e^{-2\sigma(x)}dx^2$ and $(\frac{d}{dx_i})$ forms an orthonormal basis for the tangent space at $p$, then the sectional curvature at $p$ is

$$K_p\left(\frac{d}{dx_i}, \frac{d}{dx_j}\right) = e^{2\sigma(x)}\left(\frac{d^2\sigma}{dx_i^2} + \frac{d^2\sigma}{dx_j^2} - \sum_{k\neq i,j}\left(\frac{d\sigma}{dx_k}\right)^2\right).$$

It is easy to see that if $\sigma$ is superharmonic on each two dimensional plane then the sectional curvature is bounded above by zero, and if the sectional curvature is bounded below by zero then $\sigma$ is subharmonic on each two dimensional plane (a real analog to the concept of plurisubharmonicity). In my thesis I have generalized these simple observations to conformally flat spaces of bounded curvature in the sense of Alexandrov. If $\sigma$ is superharmonic on each two dimensional plane, and $\lambda(x) = e^{-\sigma(x)}$ generates a metric, then the space has curvature bounded above by zero. For the subharmonic case a growth bound is required on $\sigma$.

Slavskii has studied conformally flat Riemannian spaces of non-negative curvature and found a converse to the condition above. Suppose that $\Phi = a + \frac{x - a}{|x - a|^2}r^2$ is a Möbius transformation on $\mathbb{R}^n$ and $\sigma_\Phi = \sigma(\Phi^{-1}(x)) + \ln|\frac{x - a}{r^2}|$. Then $(X, e^{-2\sigma_\Phi(x)}dx^2)$ is isometric to $(\Phi(X), e^{-2\sigma(x)}dx^2)$. So if $(X, e^{-2\sigma(x)}dx^2)$ has curvature bounded below by zero, so must $(\Phi(X), e^{-2\sigma_\Phi(x)}dx^2)$, and so $\sigma_\Phi$ must be subharmonic on each two dimensional plane. Slavskii has proved the following: $\sigma_\Phi$ is subharmonic on each two dimensional plane for each Möbius transformation $\Phi$ if and only if $(X, e^{-2\sigma(x)}dx^2)$ has curvature bounded below by zero.

I have been interested in generalizing this result to general spaces with curvature bounded below by zero. In my thesis I have been able to show that if the space is conformally flat with Lipschitz conformal factor, and has curvature bounded below by zero, then the induced metric on each two dimensional plane has curvature bounded below by some $K$, depending only on the Lipschitz constant and the size of the function. I conjecture that the induced subspace curvature is bounded below by zero (for a Lipschitz conformal factor). This will allow me to conclude that the result of Slavskii concerning the relationship between subharmonicity and curvature holds for this case, and from this approximation results will follow.

I also conjecture that in the non-Lipschitz case this result may not be true, and I am interested in proving this. As well as showing that the relationship between subharmonicity and curvature is does not hold in general, it will provide a new family of examples of spaces with curvature bounded below by $K$ that cannot be approximated by Riemannian spaces with curvature bounded below by $K$. 

2
Other areas of interest include determining a result parallel to that of Slavskii for spaces with curvature bounded above by zero and an analysis of the properties of general conformally flat metric spaces. I am particularly interested in finding criteria to establish whether or not given metric spaces of bounded curvature are conformally flat. Two dimensional spaces of bounded curvature are automatically conformally flat, and one can determine whether or not a Riemannian space is conformally flat by considering the Weyl tensor. If the metric is defined on a domain in $\mathbb{R}^n$ this should be a condition relating the metric geometry to the Euclidean geometry. Along with this I am interested in finding criteria to determine whether or not a given function generates a metric.

Other Research

I have previously studied entire functions, in particular looking at the growth of the real part of entire functions with P. Fenton at Otago University. This involved improving a result a W. Hayman comparing the growth of of the real part of an entire function to the growth of its maximum modulus.