Lecture 2
I. Sets
II. Sequences
III. Logic

SETS
Set: a collection of objects
Element or member: individual object in a set, \( x \in A \)

Set-builder notation: ex: \( D = \{ x \mid x \text{ is an decimal between 0 and 1} \} \)
Write in set-builder notation:
\[
\{51, 52, 53, 54, \ldots, 498, 499\}
\]
\[
\{2, 4, 6, 8, 10, \ldots\}
\]

Cardinal number of a set, \( n(X) \): indicates the number of elements in the set \( X \).
ex: If \( D = \{a, b\} \) then \( n(D) = 2 \)

Empty set or Null set: set that contains no elements; use symbol \( \emptyset \) or \( \{\} \)
note: \( \{0\} \) does not represent the empty set

Universal set: contains all elements being considered. Notation: \( U \)

Venn Diagram: named after John Venn (1834-1923) who used diagrams to illustrate ideas in logic

Complement of \( F \): set of elements in \( U \) but not in \( F \)

Subset: \( B \) is a subset of \( A \), written \( B \subseteq A \), if and only if every element of \( B \) is an element of \( A \).

Proper Subset: means that there is at least one element of \( A \) that is not in \( B \), written \( B \subset A \).

Intersection of two sets \( A \) and \( B \), written \( A \cap B \), is the set of all elements common to both \( A \) and \( B \).
\[
A \cap B = \{ x \mid x \in A \text{ and } x \in B \}
\]

Union of two sets \( A \) and \( B \), written \( A \cup B \), is the set of all elements in \( A \) or \( B \), i.e., \( A \) and \( B \) combined.
\[
A \cup B = \{ x \mid x \in A \text{ or } x \in B \}
\]

Cartesian product: written \( A \times B \), read \( A \) cross \( B \), is the set of all ordered pairs such that the first component of each pair is an element of \( A \) and the second component of each pair is an element of \( B \).
\[
A \times B = \{(x, y) \mid x \in A \text{ and } y \in B \}
\]
SEQUENCES

A Sequence is an ordered arrangement of numbers, figures, or objects.

ex 1:  0, 5, 10, 15, 20, 25, ...

ex 2:  1, 1, 2, 3, 5, 8, 13, 21, ...

ex 3:  2, 6, 10, 14, 18, 22, ...

ex 4:  1, 11, 111, 1111, 11111, ...

An arithmetic sequence is one in which each successive term is obtained from the previous term by the addition or subtraction of a fixed number, the difference. Example 1 and 3 above are arithmetic sequences.

Finding the nth term of an arithmetic sequence:

What is the 200th term of: 4, 7, 10, 13, ...?

What is the nth term of this sequence?

Find the first four terms of a sequence whose nth term is given by 2n + 3.

A geometric sequence is one in which each successive term is obtained from the previous term by multiplying the predecessor by a fixed number, the ratio.

Ex 1:  2, 4, 8, 16, 32, ...

Ex 1:  1, 3, 9, 27, 81, ...

Ex 1:  1, 1.5, 2.25, 3.375, 5.0625, ...

Can you find the 10th term of ex 1?

What is the nth term of a geometric sequence?

A Fruity Problem: A group of 10 fruit flies double in a give time period. After 10 such time periods, how many fruit flies will there be?

Music example: Do the notes of a musical scale form an arithmetic or geometric sequence?

Figurate numbers provide examples of sequences that are neither arithmetic nor geometric.

ex 1: (triangular nos.) 1, 3, 6, 10, ...

ex 2: (square nos.) 1, 4, 9, 16, ...
ex 3: (rectangular nos.) 2, 6, 12, 20, 30, ... 

Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ...
Leonardo Da Pisa, better known as Fibonacci, is considered by some scholars to have been the greatest mathematician of the Middle Ages. He published his famous treatise, *Liber abaci* (book of the abacus) in 1202. In this book Fibonacci presented the following problem:
How many pairs of rabbits will be produced in a year, beginning with a single pair if in every month each pair bears a new pair which becomes productive exactly two months after birth?

The sum of an arithmetic sequence is:
\[ S_n = \frac{n}{2}(t_1 + t_n) \]

The sum of a geometric sequence is:
\[ S_n = t_1 \cdot \frac{1-r^n}{1-r} \]

Recursion:
We observed that the Fibonacci sequence was defined by relating the \(n^{th}\) term \(u(n)\) to the two preceding terms, \(u(n-1)\) and \(u(n-2)\). Also we saw how the arithmetic and geometric sequences could be defined by relating the \(n^{th}\) term \(u(n)\) to the preceding term \(u(n-1)\). Sequences which are defined by relating the \(n^{th}\) term to one or more previous terms are said to be defined by recursion. Such sequences are very important in modeling dynamic systems such as population growth, where the population of the \(n^{th}\) generation is a function of the population of the \((n-1)^{th}\) generation.

Pill example

A set of nested squares is drawn inside a square of edge 1 unit. The corners of the next square are at the midpoints of the sides of the preceding square. The lengths of the sides form a geometric sequence. The perimeters form a geometric sequence. The areas form a geometric sequence.
1. Find the perimeter and area of the 10\(^{th}\) square.
2. The sum of the areas approach a finite number. What is it?

A snowflake curve is constructed as shown. An equilateral triangle with sides 1 unit has each side trisected. The middle sections of each side serve as bases for smaller equilateral triangles. These triangles in turn have their sides trisected, etc. As each set of smaller triangles is constructed, the total area enclosed is a partial sum of a geometric series, and the perimeter is a term in a geometric sequence.
1. Show that the area enclosed by the snowflake approaches a finite number, and tell what it is.
2. Show that the perimeter of the snowflake becomes infinite as the number of sides increases.
Suppose you just won the Urbana Sweepstakes. You have a choice of accepting a lump sum of $20000 now, or taking $100 per month for life. In either case, you will put the money in a tax-sheltered savings paying 4% interest, compounded monthly, and let the interest accumulate.

1. If you accept the $20000, the amount you have after \( n \) months is the \( n \)th term of a geometric sequence. How much will you have after 10 years?

2. If you accept the $100 per month, the amount you have after \( n \) months is the \( n \)th partial sum of a geometric series. How much will you have after 10 years?

3. How long would it be before the amount you would have from the $100 per month plan would exceed the amount from the lump sum plan?

4. Show that if you can get 7% interest compounded monthly, the $100 per month plan will never give you as much as the lump sum plan!

The IRS (Internal Revenue Service) assumes that the value of an item which can wear out decreases by a constant number of dollars each year. For example, a house "depreciates" by \( 1/40 \) of its original value each year.

What is your house worth after 1, 2, 3 years?

Do these values form an arithmetic or geometric sequence?

Why do you suppose the IRS calls this model "straight-line" depreciation?

Magic Squares:
Arrange the numbers 1-9 into a square subdivided into nine smaller squares so that the sum of every row, column, and main diagonal is the same.

The result is called a magic square.

LOGIC
Logic is a tool used in mathematical thinking and problem solving. It is essential for reasoning. In logic, a statement is a sentence that is either true or false, but not both.

ex: \( 2 + 3 = 5 \)
\( \)A hexagon has 6 sides.

These are not statements:
\( x + 3 = 5 \)
\( \)She has blue eyes.

From a given statement, it is possible to create a new statement by forming a negation. The negation of a statement is a statement with the opposite truth value of the given statement.

ex: It is snowing.

Negation: It is not snowing.
There is a symbolic system defined to help in the study of logic. If \( p \) represents a statement, the negation of the statement \( p \) is denoted by \( \neg p \) and is read "not \( p \)." Truth tables are used to show all possible true-false values for statements.

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<th>statement</th>
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<th>negation</th>
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Quantifiers: Some statements involve quantifiers and are more complicated to negate. Quantifiers include words such as all, some, every, and there exists.

- All, every, and no refer to each and every element in a set.
- Some and there exists at least one refer to one or more, or possibly all of the elements in a set.
- All, every, and each have the same mathematical meaning as do some and there exists at least one.

ex: "Some students at U of I have blue eyes" means at least one or possibly all have blue eyes.

Negation: "No students at U of I have blue eyes."

ex: "All students like hamburgers."

Negation: "Some students do not like hamburgers."

From two given statements, it is possible to create a new, compound statement by using a connective such as and or or.

Ex: It is snowing. The ski run is open.

It is snowing and the ski run is open.

This compound statement is denoted by \( p \land q \) or \( p \lor q \).

The symbols \( \land \) and \( \lor \) are used to represent and or.

"And" is called a conjunction, and "or" is called a disjunction.

Because each statement \( p \) and \( q \) may be either true or false, there are 4 distinct possibilities for the truth tables.

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<th>( p \land q )</th>
<th>( p \lor q )</th>
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Two statements are logically equivalent if, and only if, they have the same truth values. Truth tables are used to determine if two statements are logically equivalent, written \( p \leftrightarrow q \).

Statements expressed in the form "if \( p \), then \( q \)" are called conditionals, or implications, and are denoted \( p \rightarrow q \). The "if" part is called the hypothesis, and the "then" part is called the conclusion.

Many statements can be put in "if-then" form.

ex: All equilateral triangles have acute angles.

If a triangle is equilateral, then it has an acute angle.

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An implication can be worded in several different ways.

ex:  
(if p, q) If it is hot, the pool is open.
(q, if p) The pool is open if it is hot.
(p implies q) Being hot implies the pool is open.
(p only if q) Being hot is a sufficient condition for the pool to open.
(q is a necessary condition for p) Pool’s being open is a necessary condition for it to be hot.

Implications have a converse, and inverse, and a contrapositive.

ex: "If I am in Urbana, then I am in Illinois.
Converse: "If I am in Illinois, then I am in Urbana."  

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Inverse: "If I am not in Urbana, then I am not in Illinois."  

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Contrapositive: "If I am not in Illinois, then I am not in Urbana."  

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The contrapositive of a statement and the original statement have the same truth value, they are logical equivalents.

Connecting a statement and its converse with the conjunction and gives (p →q) (q →p). This compound statement is called biconditional, written p’q, and is read "p if and only if q."

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Valid reasoning: In problem solving, the reasoning is said to be valid if the conclusion follows unavoidable from true hypothesis.

Ex:  
   hypothesis: All roses are red.
   This flower is a rose.
   Conclusion: Therefore this flower is red.

The argument can be pictured with a Venn diagram.
To show that an argument is valid, all possible Venn diagrams must show that there are no contradictions. There must be no way to satisfy the hypothesis and contradict the conclusion if the argument is valid, i.e., to show an argument is not valid, you need only draw a picture that satisfies the hypothesis and contradicts the conclusion.

ex: hypothesis: All school teachers are mathematically literate.
Some mathematically literate people are not children.
Conclusion: Therefore no school teacher is a child.

A different method for determining whether an argument is valid uses direct reasoning and a form of argument called the law of detachment or modus ponens.

\[(p \rightarrow q) \quad p \rightarrow q\]

ex: If the sun is shining, then we shall take a trip.
The sun is shining.
Using the two statements, we can conclude that we shall take a trip.
In general, if the statement "if p then q" is true, then q must be true.

Another type of reasoning, indirect reasoning, uses a form of argument called modus tollens. If the conditional is true, and we know the conclusion is false, then the hypothesis must be false.

\[(p \rightarrow q) \quad \neg q \rightarrow p\]

ex: If a figure is a square, then it is a rectangle.
The figure is not a rectangle.
Conclusion: The figure cannot be a square.

Determine a conclusion for the true statements.

If \(x = 3\), then \(2x \neq 7\).
We know that \(2x = 7\).
Therefore,

The final reasoning argument to be considered involves the chain rule (also known as hypothetical syllogism).

ex: If I save, I will retire early.
If I retire early, I will become lazy.
Therefore,
In general, "if p then q" and "if q then r" are true, then "if p then r" is true.

Consider the following argument:
If a number is a power of 3, then it ends in a 1, 3, 7, or 9.
The number 3124 does not end in a 1, 3, 7, or 9.
Therefore, 3124 is not a power of 3.

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This argument is an application of modus tollens.
Draw an Venn diagram to represent the following argument and decide whether it is valid.

All timid creatures (T) are bunnies (B).
All timid creatures are furry (F).
Some cows (C) are furry.
Therefore, all cows are timid creatures.

Which of the laws (modus ponens, chain rule, or modus tollens) is being used in each of the following arguments?

1. If Joe is a professor, then he is learned. If your are learned, then you went to college. Joe is a professor, so he went to college.
2. If you have children, then you are an adult. Bob is not an adult, so he has no children.
3. If a number ends in zero, then it is a multiple of 10. Forty is a number that ends in zero, so it is a multiple of 10.