Math 115 Spring 2011

Written Homework 5 Solutions

1. Evaluate each series.
   (a) \(4 + 7 + 10 + \ldots + 55\)

Solution: We note that the associated sequence, 4, 7, 10, \ldots, 55 appears to be an arithmetic sequence. If the sequence is arithmetic, then the common difference is \(d = 7 - 4 = 10 - 7 = 3\). Then the candidate for the generating function is

\[ a(n) := 4 + (n - 1)3. \]

To prove that the associated sequence is arithmetic, we need to show that 55 is a term generated by the function \(a(n)\). That is, there is some \(N\) where \(a(N) = 55\).

\[
\begin{align*}
  a(N) &= 55 \\
  4 + (N - 1)3 &= 55 \\
  3N + 1 &= 55 \\
  3N &= 54 \\
  N &= 18
\end{align*}
\]

Since \(N = 18\) is a natural number, the series is an arithmetic series.

Using the summation formula for a finite arithmetic series,

\[
S = N \left( \frac{a_1 + a_N}{2} \right) = 18 \left( \frac{4 + 55}{2} \right) = 9(59) = 531.
\]

(b) The first and last terms of summation are 8 and \(\frac{1}{512}\), respectively and the common ratio between each term is \(\frac{1}{4}\).

Solution: Here we are given that the series is a geometric series. The generating function for the associated sequence is \(b(n) := 8 \left( \frac{1}{4} \right)^{n-1}\). For a geometric series, we have two forms of the summation formula, \(S = \frac{a_{N+1} - a_1}{r - 1}\) and \(S = a_1 \left( \frac{r^N - 1}{r - 1} \right)\) where \(N\) is the number of
terms in the series. While we could find \( N \) using the generating function, since we know the first and last terms in the series, we don’t need to. Here \( b_1 = 8 \) and \( b_N = \frac{1}{512} \).

\[
S = \frac{b_{N+1} - b_1}{r - 1} = \frac{(b_N)r - b_1}{r - 1} = \frac{(\frac{1}{512})(\frac{1}{4}) - 8}{\frac{1}{4} - 1} = \frac{\frac{1}{2048} - 8}{-\frac{3}{4}} = -\frac{4}{3} \left( \frac{1}{2048} - 8 \right)
\]
\[
= -\frac{4}{3} \left( \frac{1}{2048} - \frac{16384}{2048} \right) = -\frac{4}{3} \left( -\frac{16383}{2048} \right) = \frac{65532}{6144} = \frac{5461}{512}
\]

(c) The sum of \(-3 + 6 - 12 + 24 - \ldots\) where the associated sequence has 21 terms.

**Solution**: We note that the associated sequence, \(-3, 6, -12, 24, \ldots\) is a geometric sequence.

\[
r = \frac{6}{-3} = -\frac{12}{-6} = \frac{24}{-12} = -2
\]

The generating function for the sequence is \( c(n) := (-3)(-2)^{n-1} \). We are given that the series has 21 terms. Here we use the other formulation for the sum of a finite geometric series.

\[
S = a_1 \left( \frac{r^N - 1}{r - 1} \right) = (-3) \left( \frac{(-2)^{21} - 1}{(-2) - 1} \right) = (-2)^{21} - 1.
\]

(d) The sum of \(10, \frac{15}{2}, 5, \frac{5}{2}, \ldots\) where the associated sequence has 30 terms.

**Solution**: This is an arithmetic sequence:

\[
d = \frac{15}{2} - 10 = 5 - \frac{15}{2} = \frac{5}{2} - 5 = -\frac{5}{2}.
\]

Then \( d(n) := 10 + (n - 1) \left( -\frac{5}{2} \right) \) is the generating function for the associated sequence. We are given that \( N = 30 \). Then

\[
S = 30 \left( \frac{a_1 + a_{30}}{2} \right) = 30 \left( \frac{10 + [10 + (30 - 1) \left( -\frac{5}{2} \right)]}{2} \right) = 30 \left( \frac{20 + 29 \left( -\frac{5}{2} \right)}{2} \right)
\]
\[
= 15 \left( 20 - \frac{145}{2} \right) = 15 \left( -\frac{105}{2} \right) = -\frac{1575}{2}.
\]
2. How many terms of the sequence \(-5, -1, 3, \ldots\) must be added to give a sum of 400?

**Solution:** We need to first determine if this sequence is arithmetic or geometric. Since

\[-1 - (-5) = 3 - (-1) = 4,
\]

we assume that the sequence is arithmetic with a common difference \(d = 4\). Then, the generating function of the sequence is \(a(n) := -5 + (n - 1)(4)\). The summation formula for the associated \(N\)-term arithmetic series is

\[S = N \left( \frac{a_1 + a_N}{2} \right) = N \left( \frac{-5 + 5 + (N - 1)(4)}{2} \right).\]

We are given that the summation \(S = 400\). We need to determine \(N\).

\[
400 = N \left( \frac{-5 + 5 + (N - 1)(4)}{2} \right)
= N \left( \frac{-10 + 4N - 4}{2} \right)
= N \left( \frac{-14 + 4N}{2} \right)
= N(-7 + 2N)
= -7N + 2N^2
0 = 2N^2 - 7N - 400
\]

Factor the above quadratic equation or use the quadratic formula to solve for \(N\).

\[
N = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(-400)}}{2(2)}
= \frac{7 \pm 57}{2} = 16 \text{ or } -\frac{25}{2}.
\]

Now, both of these candidates for \(N\) can not be correct. Recall that \(N\) is a number of terms in a sequence / series. \(N\) must be a natural number. Thus,

\[N = 16\]

is the only solution.
3.

(a) Use a series to find the sum of the first 200 odd, positive integers.

Solution: We know from lecture that the sequence that generates the odd integer is odd\((n) := 1 + (n - 1)2 = -1 + 2n.\) The series here is

\[1 + 3 + 5 + \ldots + \text{odd}(200) = 1 + 3 + 5 + \ldots + [-1 + 2(100)] = 1 + 3 + 5 + \ldots + 399.\]

The summation formula for this arithmetic series is

\[S = 200 \left(\frac{1 + 399}{2}\right) = 40000.\]

(b) Use a series to find the sum of all positive integers less than 200 that are multiples of 7.

Solution: The series here is

\[7 + 14 + 21 + 28 + \ldots + 7N\]

where \(N\) is the largest natural number where \(7N \leq 200.\)

\[N \leq \frac{200}{7} = 28 + \frac{4}{7}.\]

Hence, \(N = 28.\)

We notice that the associated sequence is arithmetic: \(b(n) := 7n.\) The summation formula for this arithmetic series is

\[S = 28 \left(\frac{7 + 7(28)}{2}\right) = 14(201) = 2814.\]
4. How many terms of the sequence generated by the function \( a_n := 4(3)^{n-1} \) must be added to give a sum of 1456?

**Solution**: The associated series is clearly geometric. The summation formula for an \( N \) term geometric series is

\[
S = a_1 \left( \frac{1 - r^N}{1 - r} \right)
\]

\[
1456 = 4 \left( \frac{1 - 3^N}{1 - 3} \right)
\]

\[
\frac{364}{2} = 1 - 3^N
\]

\[
-728 = 1 - 3^N
\]

\[
-729 = -3^N
\]

\[
729 = 3^N
\]

\[
3^6 = 3^N
\]

\[
N = 6
\]

There are 6 terms in the associated series.

5. If \( 10^{a_1}, 10^{a_2}, 10^{a_3}, ..., 10^{a_n} \) is a geometric sequence, what can you determine about the sequence \( a_1, a_2, a_3, ..., a_n \)?

**Solution**: We are given that \( 10^{a_1}, 10^{a_2}, 10^{a_3}, ..., 10^{a_n} \) is a geometric sequence. Thus,

\[
r = \frac{10^{a_2}}{10^{a_1}} = \frac{10^{a_3}}{10^{a_2}} = \frac{10^{a_4}}{10^{a_3}} = \cdots = \frac{10^{a_n}}{10^{a_{n-1}}}
\]

\[
= 10^{a_2-a_1} = 10^{a_3-a_2} = 10^{a_4-a_3} = \cdots = 10^{a_n-a_{n-1}}
\]

For the numbers to all be equal, the exponent on the base 10 must be the same. That is, there is some exponent \( p \) where

\[
p = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = a_n - a_{n-1}.
\]

The equation \( a_n - a_{n-1} = p \) for all \( n \) is the definition of an arithmetic sequence with common difference \( p \). Hence, the sequence \( a_1, a_2, a_3, ..., a_n \) must be an arithmetic sequence.
6. Write \( \sum_{k=3}^{9} |2\pi - k| \) as an expanded sum and compute the sum.

**Solution:** \( \sum_{k=3}^{9} |2\pi - k| = |2\pi - 3| + |2\pi - 4| + |2\pi - 5| + |2\pi - 6| + |2\pi - 7| + |2\pi - 8| + |2\pi - 9| \).

Recall that if a number \( x \) is negative then \( |x| = -x \). Additionally, recall that \( \pi \approx 3.14159 \ldots \) Hence \( 6 < 2\pi < 7 \).

Thus, \( |2\pi - k| = 2\pi - k \) when \( k \leq 6 \) and \( |2\pi - k| = -(2\pi - k) \), when \( k > 6 \). Then,

\[
\sum_{k=3}^{9} |2\pi - k| = (2\pi - 3) + (2\pi - 4) + (2\pi - 5) + (2\pi - 6) + [(2\pi - 7)] + [-(2\pi - 8)] + [-(2\pi - 9)] \\
= 2\pi - 3 + 2\pi - 4 + 2\pi - 5 + 2\pi - 6 - (2\pi - 7) - (2\pi - 8) - (2\pi - 9) \\
= 2\pi - 3 + 2\pi - 4 + 2\pi - 5 + 2\pi - 6 - 2\pi + 7 - 2\pi + 8 - 2\pi + 9 \\
= 2\pi - 3 - 4 - 5 - 6 + 7 + 8 + 9 \\
= 2\pi + 6
\]
7. Write each of the following using summation notation.
(a) \[ a_1(b_1)^2 + a_2(b_2)^3 + a_3(b_3)^4 + \ldots + a_{10}(b_{10})^{11} \]

\textbf{Solution:} \[ \sum_{i=1}^{10} a_i(b_i)^{i+1}. \]

(b) The sum of all three digit positive even integers.

\textbf{Solution:} This is the series \(100 + 102 + 104 + \ldots + 996 + 998\). This associated sequence is arithmetic and is generated by the function \(b(n) := 100 + (n - 1)2\). Then the summation is \(\sum_{n=1}^{N} b(n)\). We need to know how many terms are in the series in order to define the upper-bound on the index in our summation notation. Note that we use \(b(N) = 998\) to determine how many terms are in the sequence.

\[ 998 = 100 + (N - 1)2 \]

\[ 898 = (N - 1)2 \]

\[ 449 = N - 1 \]

\[ N = 450 \]

Then the series can be written in the form \(\sum_{n=1}^{450} [100 + (n - 1)2]\).

Remark: An alternative method would result in the answer \(\sum_{n=50}^{499} 2n\). This is also correct.

(c) \[ 6 - 2 + \frac{2}{3} - \frac{2}{9} + \ldots + \frac{2}{243} \]

\textbf{Solution:} Again, in order to write this as a summation, we need a generating function for the associate sequence. We are lead to believe that the associate sequence is geometric because \(r = \frac{-2}{6} = -\frac{1}{3}\). To be certain that this series is a geometric series, we need to show that the term \(\frac{2}{243}\) is a term in the sequence generated by \(c(n) := 6 \left( -\frac{1}{3} \right)^{n-1}\). (If it is, in this process we will
determine how many numbers are in this sequence.)

\[
c(N) = \frac{2}{243}
\]

\[
6 \left( -\frac{1}{3} \right)^{N-1} = \frac{2}{243}
\]

\[
\left( -\frac{1}{3} \right)^{N-1} = \frac{1}{729}
\]

\[
\left( -\frac{1}{3} \right)^{N-1} = \left( -\frac{1}{3} \right)^6
\]

\[
N - 1 = 6
\]

\[
N = 7
\]

Thus,

\[
6 - 2 + \frac{2}{3} - \frac{2}{9} + \ldots + \frac{2}{243} = \sum_{n=1}^{7} 6 \left( -\frac{1}{3} \right)^{n-1}
\]
8. If \( \sum_{b=2}^{4} (a^2b - ab) = \sum_{c=3}^{5} (ac + 6) \), determine \( a \).

**Solution:** Here we expand and simplify.

\[
\sum_{b=2}^{4} (a^2b - ab) = \sum_{c=3}^{5} (ac + 6)
\]

\[
(a^22 - a2) + (a^23 - a3) + (a^24 - a4) = (a3 + 6) + (a4 + 6) + (a5 + 6)
\]

\[
2a^2 - 2a + 3a^2 - 3a + 4a^2 - 4a = 3a + 6 + 4a + 6 + 5a + 6
\]

\[
9a^2 - 9a = 12a + 18
\]

\[
9a^2 - 21a - 18 = 0
\]

\[
3(a^2 - 7a - 6) = 0
\]

\[
3(a - 3)(3a + 2) = 0
\]

Here, the summations will be equal when \( a \) is either \( a = 3 \) or \( a = -\frac{2}{3} \).
9. What is the sum of the series \( \sum_{k=1}^{n} (-1)^k \), if \( n \) is odd? if \( n \) is even?

**Solution:** Try some values for \( n \).

\[
\sum_{k=1}^{1} (-1) = -1 \\
\sum_{k=1}^{2} (-1) = (-1) + (-1)^2 = -1 + 1 = 0 \\
\sum_{k=1}^{3} (-1) = (-1) + (-1)^2 + (-1)^3 = -1 + 1 - 1 = -1 \\
\sum_{k=1}^{4} (-1) = (-1) + (-1)^2 + (-1)^3 + (-1)^4 = -1 + 1 - 1 + 1 = 0 \\
\sum_{k=1}^{5} (-1) = (-1) + (-1)^2 + (-1)^3 + (-1)^4 + (-1)^5 = -1 + 1 - 1 + 1 - 1 = -1 
\]

At this point, we recognize the pattern. If \( n \) is an even number, we can form \( n/2 \) pairs of \(-1 + 1 = 0\) and the summation will always be 0. If \( n \) is an odd number, the pairs made by the first \( n - 1 \) terms will cancel and we will be left with a single \(-1\). Hence, the summation equals -1 when \( n \) is odd.
10. 

(a) Write \( \sum_{k=1}^{10} \frac{1}{k(k+1)} \) as an expanded sum and compute the sum.

Solution:
\[
\sum_{k=1}^{10} \frac{1}{k(k+1)} = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \frac{1}{4(5)} + \frac{1}{5(6)} + \frac{1}{6(7)} + \frac{1}{7(8)} + \frac{1}{8(9)} + \frac{1}{9(10)} + \frac{1}{10(11)}
\]
\[
= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{25} + \frac{1}{30} + \frac{1}{42} + \frac{1}{72} + \frac{1}{90} + \frac{1}{110}
\]
\[
= \frac{10}{11}
\]

(b) The summation \( \sum_{k=1}^{10} \left( \frac{1}{k} - \frac{1}{k+1} \right) \) is an example of a *telescoping sum*. Expand and compute this sum. What property of the summation makes this a “telescoping sum”?

Solution:
\[
\sum_{k=1}^{10} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \left( \frac{1}{5} - \frac{1}{6} \right) + \left( \frac{1}{6} - \frac{1}{7} \right) + \left( \frac{1}{7} - \frac{1}{8} \right) + \left( \frac{1}{8} - \frac{1}{9} \right) + \left( \frac{1}{9} - \frac{1}{10} \right) + \left( \frac{1}{10} - \frac{1}{11} \right)
\]
\[
= 1 - \frac{1}{11}
\]
\[
= \frac{10}{11}
\]

The property that makes this a “telescoping sum” is the fact that it collapses down to a much smaller sum (just as a telescope can expand and collapse).

(c) Let \( N \) be a large positive number. Evaluate \( \sum_{k=1}^{N} \left( \frac{1}{k} - \frac{1}{k+1} \right) \).

Solution:
\[
\sum_{k=1}^{N} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \ldots + \left( \frac{1}{N} - \frac{1}{N+1} \right)
\]
\[
= 1 - \frac{1}{N+1} \text{ or } \frac{N}{1+N}
\]
(d) Show that \( \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)} \). 

Solution:
\[
\frac{1}{k} - \frac{1}{k+1} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{k+1-k}{k(k+1)} = \frac{1}{k(k+1)}
\]

(e) Evaluate \( \sum_{k=1}^{1000} \frac{1}{k(k+1)} \)

Solution:
\[
\sum_{k=1}^{1000} \frac{1}{k(k+1)} = \sum_{k=1}^{1000} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{1001} = \frac{1000}{1001}
\]