1. For each arithmetic sequence, find a function $a(n)$ that describes the sequence and determine the limit of each sequence.

(a) $\frac{7}{2}, 2, \frac{1}{2}, -1, ...$

**Solution:** The generating function for an arithmetic sequence has the form $a_n = 1 + (n-1)d$, where $a_1$ is the first term in the sequence and $d$ is the common difference.

Here, the first term is $a_1 = 5$. To determine $d$, we look at the difference between two successive terms.

$$\frac{7}{2} - 5 = 2 - \frac{7}{2} = \frac{1}{2} - 2 = -1 - \frac{1}{2} = -\frac{3}{2} = d$$

Thus,

$$a(n) := 5 + (n - 1) \left(-\frac{3}{2}\right)$$

For the long-run behavior, we see that we are always subtracting $-1.5$ from each successive term. Hence, the terms of this sequence will decrease without bound. We write

$$\lim_{n \to \infty} a(n) = -\infty.$$ 

(b) $-k, -k + 3, -k + 6, -k + 9, -k + 12, ...$ where $k$ is any positive number.

**Solution:** The generating function for an arithmetic sequence has the form $a_n = a_1 + (n-1)d$, where $a_1$ is the first term in the sequence and $d$ is the common difference.

Here, the first term is $a_1 = -k$. To determine $d$, we look at the difference between two successive terms.

$$-k + 3 - (-k) = -k + 6 - (-k + 3) = -k + 9 - (-k + 6) = \cdots = 3 = d$$

Thus,

$$a_n := -k + (n - 1)(3)$$

For the long-run behavior, we see that we are always adding 3 to each successive term. Hence, the terms of this sequence will increase without bound. We write

$$\lim_{n \to \infty} a_n = \infty.$$
Note that the value of $k$ plays no role in the long-run behavior. No matter how large $k$ is, there will eventually be a multiple of 3 what will turn the term $-k + (n - 1)(3)$ into a positive number. (We just need to choose $n$ large enough.)
2. Consider an arithmetic sequence whose fourth term is 41 and whose ninth term is 85. Find the function \( f(n) \) that describes this sequence and find the 20th term of the sequence.

\textbf{Solution}: This is identical to an example from lecture. The following is “a” solution. (There is more than one way to do this problem.)

Since the sequence is arithmetic, we know the following:

\[ a_{n+1} = a_n + d \text{ from the definition,} \]

\[ a_n := a_1 + (n - 1)d \text{ the generating function.} \]

We are given that \( a_4 = 41 \) and \( a_9 = 85 \). Using the common difference \( d \),

\[ a_9 = a_4 + d + d + d + d + d. \]

\[ 85 = 41 + 5d. \]

\[ d = \frac{44}{5}. \]

To find \( a_1 \), we use the generating function now that we know \( d \).

\[ a_4 = a_1 + (4 - 1)d = a_1 + 3d. \]

\[ 41 = a_1 + 3\left(\frac{44}{5}\right). \]

\[ a_1 = \frac{73}{5}. \]

Hence

\[ a_n := \frac{73}{5} + (n - 1) \left(\frac{44}{5}\right). \]

To find the 20th term, \( a_{20} = \frac{73}{5} + (20 - 1) \left(\frac{44}{5}\right) = \frac{909}{5}. \)
3. Determine the function that generates the sequence 13, 21, 29, 37, ....157 and determine how many terms are in the sequence.

**Solution:** Again, this is an arithmetic sequence. We see that by showing that there is a common difference between the first four terms.

\[ 21 - 13 = 29 - 21 = 37 - 29 = 8 = d. \]

Since \( a(1) = 13 \), the generating function is

\[ a(n) : 13 + (n - 1)8 \]

If 157 is a term of this arithmetic sequence, there must be some integer \( \alpha \) such that \( a(\alpha) = 157 \). Thus,

\[ 157 = 13 + (\alpha - 1)8 \]

must have an integer solution. Solving the equation for \( \alpha \),

\[ \alpha = \frac{157 - 13}{8} + 1 = 19. \]

Thus, there are 19 terms in this sequence.
4. For each geometric sequence, find a function \( a(n) \) that describes the sequence and determine the limit of each sequence.

(a) \( \frac{3}{64}, -\frac{3}{16}, \frac{3}{4}, -3... \)

**Solution:** The general form of the generating function for a geometric sequence is given by

\[ a_n := a_1 r^{n-1}. \]

Here, \( a_1 = \frac{3}{64} \). To find \( r \), we need to look at the ratio of any two successive terms; \( r = \frac{a_{n+1}}{a_n} \).

\[ r = \frac{a_2}{a_1} = \frac{-\frac{3}{16}}{\frac{3}{64}} = -\frac{3}{16} \cdot \frac{64}{3} = -4. \]

Then,

\[ a_n := \left( \frac{3}{64} \right) (-4)^{n-1}. \]

The limit of this sequence does not exist because \( r \leq -1 \).

(b) \( 16, 8, 4, 2, ... \)

**Solution:**

Here, \( a_1 = 16 \). To find \( r \), we need to look at the ratio of any two successive terms;

\[ r = \frac{a_{n+1}}{a_n}. \]

\[ r = \frac{a_2}{a_1} = \frac{8}{16} = \frac{1}{2}. \]

Then,

\[ a_n := 16 \left( \frac{1}{2} \right)^{n-1}. \]

The limit of this sequence is 0 because \( -1 \leq r \leq 1 \).
5. Consider a sequence whose 4th term is 7 and whose 9th term is 224.

(a) Assume the sequence is arithmetic. Find the function \( a_n \) that describes the sequence and find the 21st term of the sequence.

**Solution:** The general form of the generating function for an arithmetic sequence is given by

\[
a_n := a_1 + (n - 1)d.
\]

Using the given information \( a_4 = 7 \) and \( a_9 = 224 \). There are MANY ways to solve for \( a_1 \) and \( d \) given this information; here is one method:

\[
a_4 = a_1 + (4 + 1)d = 7 \quad \rightarrow \quad a_1 + 3d = 7
\]

\[
a_9 = a_1 + (9 + 1)d = 224 \quad \rightarrow \quad a_1 + 8d = 224
\]

Since \( a_1 = 7 - 3d \),

\[
(7 - 3d) + 8d = 224
\]

\[
7 + 5d = 224
\]

\[
5d = 217
\]

\[
d = \frac{217}{5}.
\]

Then \( a_1 = 7 - 3 \left( \frac{217}{5} \right) = -\frac{616}{5} \).

\[
a_n := -\frac{616}{5} + (n - 1) \left( \frac{217}{5} \right)
\]

The 21st term in the sequence is \( a_{21} = -\frac{616}{5} + (21 - 1) \left( \frac{217}{5} \right) = \frac{3724}{5} \).
(b) Assume the sequence is geometric. Find the function $a_n$ that describes the sequence and find the 21st term of the sequence.

**Solution**: The general form of the generating function for a arithmetic sequence is given by

$$a_n := a_1 r^{n-1}.$$ 

Using the given information $a_4 = 7$ and $a_9 = 224$.

$$a_4 = a_1 r^{4-1} = 7 \quad \rightarrow \quad a_1 r^3 = 7$$

$$a_9 = a_1 r^{9-1} = 224 \quad \rightarrow \quad a_1 r^8 = 224$$

Since $a_1 = \frac{7}{r^3}$,

$$\left(\frac{7}{r^3}\right) r^8 = 224$$

$$r^5 = \frac{224}{7}$$

$$r^5 = 32$$

$$r = \sqrt[5]{32}$$

$$r = 2.$$ 

Then $a_1 = \frac{7}{2^3} = \frac{7}{8}$

$$a_n := \left(\frac{7}{8}\right) 2^{n-1}$$

The 21st term in the sequence is $a_{21} = \left(\frac{7}{8}\right) 2^{21-1} = \left(\frac{7}{8}\right) 2^{20}$ or $7(2)^{17}$. 

6. Determine if the following sequences are arithmetic, geometric, or neither and find a function \( a(n) \) that generates the sequence. Then determine the limit of each sequence.

(a) \( 1, -\sqrt{3}, 3, -3\sqrt{3}, ... \)

Solution: Here

\[
a_1 = 1, a_2 = -\sqrt{3}, a_3 = 3, \text{ and } a_4 = -3\sqrt{3}.
\]

If the sequence is arithmetic,

\[
d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3.
\]

\[
a_2 - a_1 = -\sqrt{3} - 1
\]

\[
a_3 - a_2 = 3 - (-\sqrt{3}) = 3 + \sqrt{3} \neq a_2 - a_1.
\]

Thus, this sequence is not an arithmetic sequence.

If the sequence is geometric,

\[
\begin{align*}
r &= \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3}. \\
\frac{a_2}{a_1} &= \frac{-\sqrt{3}}{1} = -\sqrt{3} \\
\frac{a_3}{a_2} &= \frac{3}{-\sqrt{3}} = \frac{3\sqrt{3}}{-3} = -\sqrt{3} \\
\frac{a_4}{a_3} &= \frac{-3\sqrt{3}}{3} = -\sqrt{3}
\end{align*}
\]

Hence \( r = -\sqrt{3} \) and the sequence is geometric.

\[
a_n := (1)(-\sqrt{3})^{n-1}
\]

The limit of this sequence does not exist since \( r = -\sqrt{3} \leq -1 \).

(b) \(-5, -\frac{5}{2}, 0, \frac{5}{2}, ... \)

Solution: Here

\[
a_1 = -5, a_2 = -\frac{5}{2}, a_3 = 0, \text{ and } a_4 = \frac{5}{2}.
\]
If the sequence is arithmetic,

\[ d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3. \]

\[
\begin{align*}
a_2 - a_1 &= -\frac{5}{2} - (-5) = \frac{5}{2} \\
a_3 - a_2 &= 0 - \left( -\frac{5}{2} \right) = \frac{5}{2} \\
a_4 - a_3 &= \frac{5}{2} - 0 = \frac{5}{2}.
\end{align*}
\]

Hence the sequence is arithmetic and there is a common difference \( d = \frac{5}{2}. \)

\[ a_n := -5 + (n - 1) \left( \frac{5}{2} \right) \]

Since \( d > 0, \) the terms of this sequence increase without bound.

\[ \lim_{n \to \infty} a_n = \infty. \]
7. Consider the infinite sequence \( k^2, 4k^3, 16k^4, 64k^5, \ldots \).

(a) Determine the function that generates this geometric sequence.

**Solution:** Here \( a_1 = k^2 \).

\[
r = \frac{a_2}{a_1} = \frac{4k^3}{k^2} = 4k
\]

Then

\[
a_n := k^2 (4k)^{n-1}.
\]

(b) Determine the limit of this sequence. (Warning: Your answer must be dependent on the constant \( k \).)

**Solution:** The limit of a geometric sequence exists when the common ratio \( r \) is between \(-1\) and 1 or \( r = 1 \). In other words,

\[-1 < r \leq 1\]

Here, \( r = 4k \) requires

\[-1 < 4k \leq 1\]

\[-\frac{1}{4} < k \leq \frac{1}{4}\]

Thus, the limit of this geometric sequence exists when \( k \in \left(-\frac{1}{4}, \frac{1}{4}\right) \). Another way to write this is

\[
\lim_{n \to \infty} a_n = \begin{cases} 
0 & \text{if } -\frac{1}{4} < k < \frac{1}{4} \\
\frac{1}{16} & \text{if } k = \frac{1}{4} \\
\infty & \text{if } k > \frac{1}{4} \\
does not exist & \text{if } k \leq -\frac{1}{4}
\end{cases}
\]
8. Prove that the following sequences are neither arithmetic nor geometric. Then find the next 4 terms of each sequence.

(a) 5, 6, 8, 11, 15, ...

**Solution:** Here

\[ a_1 = 5, a_2 = 6, a_3 = 8, a_4 = 11 \text{ and } a_5 = 15. \]

If the sequence is arithmetic,

\[ d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = a_5 - a_4. \]

\[ a_2 - a_1 = 6 - 5 = 1 \]

\[ a_3 - a_2 = 8 - 6 = 2 \neq a_2 - a_1 \]

Thus, this sequence is not an arithmetic sequence.

If the sequence is geometric,

\[ r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \frac{a_5}{a_4}. \]

\[ \frac{a_2}{a_1} = \frac{6}{5} \]

\[ \frac{a_3}{a_2} = \frac{8}{6} \neq \frac{a_2}{a_1} \]

Thus, this sequence is not a geometric sequence.

Here, to find the next four terms, we need to uncover the pattern. Notice that the difference between the first two terms is 1. The difference between the second two terms is 2. The difference between the third two terms is 3, etc. That is,

\[ \begin{align*}
    a_2 &= a_1 + 1 \\
    a_3 &= a_2 + 2 \\
    a_4 &= a_3 + 3 \\
    a_5 &= a_4 + 4
\end{align*} \]
Hence,

\[ a_6 = a_5 + 5 = 15 + 5 = 20 \]
\[ a_7 = a_6 + 6 = 20 + 6 = 26 \]
\[ a_8 = a_7 + 7 = 26 + 7 = 33 \]
\[ a_9 = a_8 + 8 = 33 + 8 = 41 \]

(b) 100, 50, 10, 5, 1, ...

**Solution:** Here

\[ a_1 = 100, a_2 = 50, a_3 = 10, a_4 = 5 \text{ and } a_5 = 1. \]

If the sequence is arithmetic,

\[ d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = a_5 - a_4. \]
\[ a_2 - a_1 = 50 - 100 = -50 \]
\[ a_3 - a_2 = 10 - 50 = -40 \neq a_2 - a_1 \]

Thus, this sequence is not an arithmetic sequence.

If the sequence is geometric,

\[ r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \frac{a_5}{a_4}. \]
\[ \frac{a_2}{a_1} = \frac{50}{100} = \frac{1}{2} \]
\[ \frac{a_3}{a_2} = \frac{10}{50} = \frac{1}{5} \neq \frac{a_2}{a_1} \]

Thus, this sequence is not a geometric sequence.

Here, to find the next four terms, we need to uncover the pattern. Notice that the ratio pattern discovered in showing that the sequence is not geometric continues. That is,

\[ \frac{a_4}{a_3} = \frac{5}{10} = \frac{1}{2} \]
\[
a_5 = \frac{1}{5} = \frac{1}{5}
\]

Hence, to find \(a_6\), we multiply \(a_5\) by \(1/2\).

\[
a_6 = a_5(1/2) = (1)(1/2)
\]

To find \(a_7\), we multiply \(a_6\) by \(1/5\).

\[
a_7 = \frac{1}{2} \left( \frac{1}{5} \right) = \frac{1}{10}
\]

Continuing the pattern,

\[
a_8 = \frac{1}{10} \left( \frac{1}{2} \right) = \frac{1}{20}
\]

\[
a_9 = \frac{1}{20} \left( \frac{1}{5} \right) = \frac{1}{100}
\]
9. For what value(s) of $k$ will $k - 4, k - 1, 2k - 2$ be a geometric sequence?

**Solution:** Here

$$a_1 = k - 4, a_2 = k - 1, \text{ and } a_3 = 2k - 2.$$

If the sequence is geometric,

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2}.$$

\[\frac{a_2}{a_1} = \frac{a_3}{a_2} \rightarrow \frac{k - 1}{k - 4} = \frac{2k - 2}{k - 1}.\]

**Note 1:** This equation reminds us of an important fact about geometric sequences. A geometric sequence can not have a single term that is 0. [ ]

\[a_1 = k - 4 \neq 0 \rightarrow k \neq 4\]

\[a_2 = k - 1 \neq 0 \rightarrow k \neq 1\]

\[a_3 = 2k - 2 \neq 0 \rightarrow k \neq 1 \text{ (again)}\]

Back to the problem at hand:

\[\frac{k - 1}{k - 4} = \frac{2k - 2}{k - 1}.\]

\[(k - 1)(k - 1) = (2k - 2)(k - 4)\]

\[(k - 1)(k - 1) = 2(k - 1)(k - 4)\]

\[(k - 1) = 2(k - 4)\]

\[k - 1 = 2k - 8\]

\[7 = k\]

**Note 2:** Foolishly, this question forgets to ask for the generating function. If it did, use $k = 7$ to find $a_1$ and $r$.

\[a_1 = (7) - 4 = 3\]

\[r = \frac{a_2}{a_1} = \frac{7 - 1}{3} = 2\]

Then, the generating function for this sequence is

\[a(n) := 3(2)^{n-1}.\]