\[ 6.8 \]

a.) \[ 224 = 1 \cdot 126 + 98 \]
\[ 126 = 1 \cdot 98 + 28 \]
\[ 98 = 3 \cdot 28 + 14 \rightarrow \text{gcd} = 14 \]
\[ 28 = 2 \cdot 14 + 0 \]

\[ 14 = 98 - 3 \cdot 28 \]
\[ = 98 - 3 \cdot (126 - 1 \cdot 98) \]
\[ = 4 \cdot 98 - 3 \cdot 126 \]
\[ = 4 \cdot (224 - 1 \cdot 126) - 3 \cdot 126 \]
\[ = 4 \cdot 224 - 7 \cdot 126 \]

b.) \[ 299 = 1 \cdot 221 + 78 \]
\[ 221 = 2 \cdot 78 + 65 \]
\[ 78 = 1 \cdot 65 + 13 \rightarrow \text{gcd} = 13 \]
\[ 65 = 5 \cdot 13 + 0 \]

\[ 13 = 78 - 65 \]
\[ = 78 - (221 - 2 \cdot 78) \]
\[ = 3 \cdot 78 - 221 \]
\[ = 3 \cdot (299 - 1 \cdot 221) - 221 \]
\[ = 3 \cdot 299 - 4 \cdot 221 \]
6.9

a.) Solve \(17x + 13y = 1\).

\[
17 = 1 \cdot 13 + 4
\]
\[
13 = 3 \cdot 4 + 1 \quad \Rightarrow \quad 1 = 13 - 3 \cdot 4
\]
\[
y = 4 \cdot 1 + 0 = 4 \cdot 13 - 3 \cdot 17
\]
\[
\therefore \quad (x, y) = (-3, 4)
\]

Now scale by 200 to find a solution to \(17x + 13y = 200\):

\[
(x, y) = (-600, 800)
\]

\[
\text{slope} \quad \frac{-17}{13}
\]

All solutions:
\[
\{(600 + 13t, 800 - 17t) \mid t \in \mathbb{Z}\}
\]

b.) \(\gcd(21, 15) = 3\) so divide by 3:

\[
7x + 5y = 31
\]

Solve \(7x + 5y = 1\). By inspection, \((-2, 3)\) is a solution. (Or use Euclidean algorithm.) Scale by 31:

\[
(-62, 93) \text{ is a solution to } 7x + 5y = 31.
\]

All solutions:
\[
\{(62 + 5t, 93 - 7t) \mid t \in \mathbb{Z}\}
\]
c.) No solutions: \( \gcd(60, 42) = 6 \), and \( 6 \nmid 104 \).
d.) \( \gcd(588, 231) ? \)

\[
\begin{align*}
588 &= 2 \cdot 231 + 126 \\
231 &= 1 \cdot 126 + 105 \\
126 &= 1 \cdot 105 + 21 \leftarrow \gcd = 21 \\
105 &= 5 \cdot 21 + 0
\end{align*}
\]

21 \| 63, so there are solutions. Divide equation by 21:

\[
28x + 11y = 3
\]

Find a solution to \( 28x + 11y = 1 \) ... by inspection, \( (2, -5) \) is a solution. Scale by 3 to get \( (x, y) = (6, -15) \).

All solutions: \( \{ (6 + 11t, -15 - 28t) \mid t \in \mathbb{Z} \} \)

---

\boxed{6.28} Suppose the prime factorization of \( a \) is \( \prod_{i=1}^{r \text{ of } n} p_i^{e_i} \) and the prime factorization of \( b \) is \( \prod_{i=1}^{r \text{ of } n} q_i^{d_i} \); then no \( p_i \) is equal to any \( q_j \), since \( \gcd(a, b) = 1 \). Now, if \( a \mid n \), then each \( p_i \) of \( n \) with multiplicity at least \( e_i \). Likewise, since \( b \mid n \), each of \( q_j \)'s must occur in the prime factorization of \( n \), with multiplicity at least \( d_j \). Thus \( n \) is of the form \( \prod_{i=1}^{r \text{ of } n} p_i^{e_i} \prod_{i=1}^{r \text{ of } n} q_i^{d_i} \) with \( E_i \leq e_i \), \( D_i \geq d_i \) for all \( i \). This is clearly divisible by \( ab = \prod_{i=1}^{r \text{ of } n} p_i^{e_i} \prod_{i=1}^{r \text{ of } n} q_i^{d_i} \).
6.29] If \( a = \prod_{i=1}^{\infty} p_i^{e_i} \) and \( b = \prod_{i=1}^{\infty} p_i^{d_i} \), then
\[
\text{gcd}(a, b) = \prod_{i=1}^{\infty} p_i^{\min(e_i, d_i)}
\]
and
\[
\text{lcm}(a, b) = \prod_{i=1}^{\infty} p_i^{\max(e_i, d_i)}.
\]
Now, \( ab = \prod_{i=1}^{\infty} p_i^{e_i+d_i} \), while \( \text{lcm}(a, b) \cdot \text{gcd}(a, b) = \prod_{i=1}^{\infty} p_i^{\max(e_i+d_i)} \)
\[
= \prod_{i=1}^{\infty} p_i^{e_i+d_i}
\]
as well. Thus \( ab = \text{lcm}(a, b) \cdot \text{gcd}(a, b) \).

6.34] Suppose by contradiction that it is finite, and that all the primes are \( p_1, \ldots, p_n \). Consider the number \( p_1 p_2 \cdots p_n + 1 \). It is divisible by some prime number, but not by any of \( p_1, \ldots, p_n \). So there must be some other prime number which is \( p_k, \ldots, p_n \). So there must be some other prime number which is not on our list, contradicting the fact that we had listed them all.

6.47] Multiply through by 60: \( 1 = 12x + 5y \).
One solution (by inspection): \((2, 5)\).
All solutions: \( \{(-2+5t, 5-12t) \mid t \in \mathbb{Z}\} \).
6.48 We proved in class (see Theorem 6.12) that the set of $\mathbb{Z}$-linear combinations of $a, b$ is precisely the set of all multiples of $\gcd(a, b)$. Thus if $c$ is a multiple of $d$, it is expressible as a linear combination of $a, b$, i.e., the equation $ax + by = c$ has a solution.

The general pattern of these problems is as follows: Divide by $d$ to get
\[
\begin{align*}
\frac{a}{d} x + \frac{b}{d} y &= \frac{c}{d} \\
\end{align*}
\]
then find a solution $(x_0, y_0)$. Then all solutions are of the form
\[
\mathcal{Z} \{ (x_0 + \frac{b}{d} t, y_0 - \frac{a}{d} t) \mid t \in \mathbb{Z} \}.
\]

6.55 Let $I \subseteq \mathbb{Z}$ be an ideal. If $I = (0)$, then $I$ is principal, so assume $I \neq (0)$. Then $I$ contains some positive numbers, hence it contains a smallest positive number, say $n$. Then we claim that $I = (n)$, i.e., every element of $I$ is a multiple of $n$. Suppose not, i.e., suppose that there is some $b \in I$ which is not a multiple of $n$. Then we can divide $b$ by $n$, to get $b = qn + r$ with $0 < r < n$. But then $r = b - qn \in I$ (since $b \in I$ and $n \in I$), and it is smaller than $n$. This contradicts our choice of $n$ as the smallest positive element of $I$. 

\[
(6.56) \frac{3x^2}{x^3 + x + 1}
\]
\[
\frac{3x^3}{x + 1}
\]
\[
\frac{x - 1}{x + 1}
\]
\[
\frac{x^2}{x^2 + x}
\]
\[
\frac{-x}{x - 1}
\]
\[
\frac{-x - 1}{1}
\]
\[
\frac{x + 1}{x + 1}
\]
\[
\frac{x}{x}
\]
\[
\frac{1}{0}
\]

\text{gcd is 1.}
b.) \[
\frac{x^2 + x}{x^3 + 2x^2 + 2x + 1} \div \frac{x^3 + x^2}{x^2 + 2x + 1} \div \frac{x^2 + x}{x + 1}
\]
\[
\begin{array}{c|c}
 & \hline \\
x + 1 & x^2 + 1 \\
\hline
x + 1 & x^2 + x \\
\hline
0 & \\
\end{array}
\]
gcd is \(x + 1\).

c.) \[
\frac{x^3 - 3x - 2}{x^3 - 2x^2 - x + 2} \div \frac{x^3 - 3x - 2}{-2x^2 + 2x + 4} \leq \text{Make monic!} \\
\text{Divide by } -2.
\]
\[
\begin{array}{c|c}
 & \hline \\
x^2 - x + 2 & x + 1 \\
\hline
x^2 - x + 2 & -3x - 2 \\
\hline
0 & \\
\end{array}
\]
gcd is \(x^2 - x - 2\).
6.6.1 The ideal

\[(x, y) = \{ p(x, y)x + q(x, y)y \mid p, q \in \mathbb{R}[x, y] \} \]

is not principal. Sketch of proof: If it had a lone generator, that generator would have to be either \(x\) or \(y\). But \(xy, yx\), and \(yxy\). So this isn’t possible.

(This problem will not be graded.)