Train Tracks and Applications, Part III

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Outline

1. Part I: Train tracks for irreducible automorphisms: definitions, train track algorithm, application to surface homeomorphisms (done)

2. Part II: Train tracks for reducible automorphisms: relative train track maps, improved relative train track maps, actions on R-trees, Scott conjecture (done)

3. Part III: Hyperbolic automorphisms: mapping tori of free group automorphisms, necessary and sufficient conditions for hyperbolicity

Outline of Part III

- Introduction: Statement of results, history
- Prerequisites: Nielsen paths, improved train track maps
- Main ideas: Measuring forward and backward growth
- Proof of the main theorem

References

- Bestvina-Feighn-Handel: The Tits alternative for Out(F_n) I, to appear in *Annals of Mathematics*

- Bestvina-Feighn: A combination theorem for negatively curved groups, *Journal of Differential Geometry,* 1992

- B.: Hyperbolic automorphisms of free groups, to appear in *Geometric and Functional Analysis*

- B.: Mapping tori of automorphisms of hyperbolic groups, PhD thesis

The main definition

Let $F =< x_1, \ldots, x_n >$ be a finitely generated free group.

Definition. The *mapping torus* of an automorphism ϕ is the semidirect product

$$M_\phi =< x_1, \ldots, x_n, t \mid txt^{-1} = \phi(x) > .$$

An automorphism $\phi : F \to F$ is called *hyperbolic* if there exist numbers $\lambda > 1$ and $M > 0$ such that

$$\lambda |x| \leq \max\{|\phi^N(x)|, |\phi^{-N}(x)|\}$$

for all $x \in F$.

An automorphism $\phi : F \to F$ is called *atoroidal* if it has no nontrivial periodic conjugacy classes. Observe that the mapping torus of such an automorphism contains no subgroups isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

The main result

Theorem (B.). *If ϕ is an atoroidal automorphism of F, then ϕ is hyperbolic.*

This theorem completes the proof of the following statement:

Theorem (Bestvina-Feighn,Gersten,B.). *The following statements are equivalent.*

1. *The mapping torus $F \rtimes_\phi \mathbb{Z}$ of ϕ is hyperbolic.*

2. *ϕ is hyperbolic.*

3. *ϕ is atoroidal.*

Note: A corresponding result holds for automorphisms of hyperbolic groups.

The geometric setting

Lemma. *Let $\phi : F \to F$ be an atoroidal automorphism. Then ϕ is hyperbolic if there exist numbers $M > 0$ and $\lambda > 1$ such that*

$$\lambda ||g|| \leq \max\{||\phi^M(g)||, ||\phi^{-M}(g)||\}$$

for all $g \in F$, where $||.||$ denotes the length of the shortest word in the conjugacy class of g.

If $f : G \to G$ is a homotopy equivalence of a finite graph representing the outer automorphism determined by ϕ, then there exists a biLipschitz equivalence between the lengths of loops in G and the lengths of the corresponding conjugacy classes in F, so we can detect hyperbolicity by studying f.

We call a homotopy equivalence *hyperbolic* if it represents a hyperbolic automorphism.

Relative train track maps

A topological representative $f : G \to G$ is a *relative train track map* if the following conditions are satisfied for all exponentially growing strata H_r:

1. If E is an edge in H_r, then the first and last edges of the edge path $f(E)$ lie in H_r. In particular, if (E_1, E_2) is a turn with E_1 in G_r and E_2 in G_{r-1}, then (E_1, E_2) is legal (read: mixed turns are legal).

2. If ρ is a nontrivial path in G_{r-1} with endpoints in $H_r \cap G_{r-1}$, then $f(\rho)$ is homotopic (rel endpoints) to a nontrivial path with endpoints in $H_r \cap G_{r-1}$.

3. If E is an edge in H_r, then $f(E)$ contains no illegal turns in H_r (read: $f(E)$ is r-legal).

Nielsen paths

A path ρ in G is said to be a *(periodic) Nielsen path* if ρ is not constant and if $[f^k(\rho)] = \rho$ for some $k > 0$. The *period* of ρ is the smallest such exponent k.

A *pre-Nielsen* path is a path whose image under some iterate of f is a Nielsen path. A Nielsen path is called *indivisible* if it cannot be written as a concatenation of two Nielsen paths.

A decomposition of a path or circuit $\sigma = \sigma_1 \cdots \sigma_s$ into subpaths is called a *splitting* if $[f^k(\sigma_i)] = [f^k(\sigma_i)]_\sigma$ for all k, i.

Notation: $[\rho]$ denotes a path tightened relative endpoints, and $[\rho]_\sigma$ denotes the path $[\rho] \cap \sigma$.

Improved relative train track maps

Theorem. *For every atoroidal outer automorphism $\mathcal{O}$ of F, there exists an exponent $k > 0$ such that $\mathcal{O}^k$ is represented by a relative train track map $f : G \to G$ with the following additional properties:*

1. *Every periodic Nielsen path has period one.*

2. *If H_r is a polynomially growing stratum, then H_r consists of a single edge E_r, and $f(E_r) = E_r u_r$, where $u_r \subset G_{r-1}$.*

3. *If H_r is an exponentially growing stratum, then there is at most one indivisible Nielsen path ρ in G_r that intersects H_r nontrivially, and at least one of the endpoints of ρ is not contained in $H_r \cap G_{r-1}$.*

We call f an *improved* relative train track map.

The intuition behind the proof

- Although the definition of hyperbolicity involves both positive and negative iterates of the automorphism ϕ, we run the entire proof in just *one* relative train track map $f : G \to G$ representing some power of ϕ.

- We use two different ways of measuring the length of a circuit σ in G, namely the length of legal subpaths of σ and the number of illegal turns in σ.

- Roughly speaking, the length of sufficiently long legal subpaths grows exponentially under forward iteration, while the number of illegal turns grows exponentially under backward iteration.

A technical result

Proposition. *For all $L > 0$, there exists some exponent $M > 0$ such that if ρ is a path in G_r with $L_r(\rho) \geq 1$, one of the following three statements holds:*

1. *$[f^M(\rho)]$ has an r-legal segment of r-length greater than L.*

2. *$[f^M(\rho)]$ has fewer r-illegal turns than ρ.*

3. *ρ can be expressed as a concatenation $\tau_1\rho'\tau_2$, where $L_r(\tau_1) \leq 2L$, $L_r(\tau_2) \leq 2L$, $i_r(\tau_1) \leq 1$, $i_r(\tau_2) \leq 1$, and ρ' splits as a concatenation of pre-Nielsen paths (with one r-illegal turn each) and segments in G_{r-1}.*

Lemma (from Ramsey theory). *For all natural numbers K, N_0, Q there exists some M such that for all maps $f : \{1, \cdots, M\} \to \{1, \cdots, K\}$ there exist numbers n and $N \geq N_0$ such that $f(n) = f(n+N) = \cdots = f(n+QN)$.*

The key to induction

Lemma. *Let $f : G \to G$ be a homotopy equivalence, and let G' be a subgraph of G such that the restriction of f to G' is a hyperbolic homotopy equivalence of G'.*

Then there exist constants $L^C, \lambda > 1$ and N with the property that, if $\rho \subset G'$ is a subpath of some circuit σ in G and if the length of ρ is at least L^C, we have

$$\lambda L(\rho) \leq \max\{L([f^N(\rho)]_\sigma), L(\alpha)\},$$

where α is some subpath of σ^{-N} satisfying $[f^N(\alpha)]_{(\sigma^{-N})} = \rho$.