Train Tracks and Applications, Part II(b)

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Outline

1. Part I: Train tracks for irreducible automorphisms: definitions, train track algorithm, application to surface homeomorphisms (done)

2. Part II: Train tracks for reducible automorphisms: relative train track maps, actions on R-trees, Scott conjecture

3. Part III: Hyperbolic automorphisms: mapping tori of free group automorphisms, necessary and sufficient conditions for hyperbolicity

Outline of Part II(b)

- Scott conjecture

- invariant actions on $\mathbb{R}$-trees

- dendrological proof of the Scott conjecture (sketch)

References

- Bestvina: $\mathbb{R}$-trees in topology, geometry, and group theory, preprint

- Gaboriau-Levitt-Lustig: A dendrological proof of the Scott conjecture for automorphisms of free groups, *Proc Edinburgh Math. Soc.*, 1998

- Gaboriau-Jaeger-Levitt-Lustig: An index for counting fixed points of automorphisms of free groups, *Duke Math. J.*, 1998

Scott conjecture

Theorem (Scott conjecture). *Let α be an automorphism of a free group of rank n. Then the fixed subgroup*

$$\mathit{Fix}(\alpha) = \{g \in F \mid \alpha(g) = g\}$$

has rank at most n.

History:

- Gersten first proved that Fix(α) is finitely generated.
- Cooper was the first to publish a proof of the same fact.
- Bestvina-Handel gave the first proof of the Scott conjecture.
- Gaboriau-Levitt-Lustig shortened the proof by using $\mathbb{R}$-tree techniques.

$\mathbb{R}$-trees (definition and examples)

Definition. A metric space X is called an $\mathbb{R}$-*tree* if for any two points $x, y \in X$, there is a unique arc from x to y and this arc is a geodesic segment.

Examples:

1. simplicial trees

2. $\mathbb{R}^2$ with the SNCF metric (French railway metric)

3. $\mathbb{R}^2$ with the 'infinite comb' metric

Actions on $\mathbb{R}$-trees

- Given an $\mathbb{R}$-tree T with an action (by isometries) of a free group F, we define a *length function* $l : F \to \mathbb{R}^+$, where

$$l(w) = \inf_{x \in T} \{d(x, wx)\}.$$

- An isometry h of an $\mathbb{R}$-tree T is either *elliptic* (i.e., h fixes a point in T) or *hyperbolic* (i.e., there is a line in T on which h acts by translation).

- An action of F on T is *nontrivial* if there is no global fixed point.

- An action of F on T is *minimal* if there is no proper invariant subtree.

Invariant actions on $\mathbb{R}$-trees

A *homothety* $H : T \to T$ *with stretching factor* λ is a map satisfying

$$d(H(x), H(y)) = \lambda d(x, y)$$

for all $x, y \in T$.

Theorem. *Given an automorphism* α *of* F, *there exists a nontrivial minimal action of* F *on an* $\mathbb{R}$*-tree* T, *with trivial edge stabilizers, whose length function* l *satisfies* $l \circ \alpha = \lambda l$ *for some* $\lambda \geq 1$. *There exists a homothety* $H : T \to T$ *with stretching factor* λ *such that*

$$\alpha(w)H = Hw$$

for all $w \in F$. *If* $\lambda = 1$, T *may be taken to be a simplicial tree.*

Moreover, the rank of a stabilizers of a point is at most $n - 1$, *where* n *is the rank of* F.

Relative train track maps

Recall: A topological representative $f : G \to G$ is a *relative train track map* if the following conditions are satisfied for all exponentially growing strata H_r:

1. If E is an edge in H_r, then the first and last edges of the edge path $f(E)$ lie in H_r. In particular, if (E_1, E_2) is a turn with E_1 in G_r and E_2 in G_{r-1}, then (E_1, E_2) is legal (read: mixed turns are legal).

2. If ρ is a nontrivial path in G_{r-1} with endpoints in $H_r \cap G_{r-1}$, then $f(\rho)$ is homotopic (rel endpoints) to a nontrivial path with endpoints in $H_r \cap G_{r-1}$.

3. If E is an edge in H_r, then $f(E)$ contains no illegal turns in H_r (read: $f(E)$ is r-legal).