Train Tracks and Applications, Part II

Peter Brinkmann

September 14, 2000

Outline

1. Part I: Train tracks for irreducible automorphisms: definitions, train track algorithm, application to surface homeomorphisms (done)

2. Part II: Train tracks for reducible automorphisms: relative train track maps, improved relative train track maps, actions on R-trees, Scott conjecture

3. Part III: Hyperbolic automorphisms: mapping tori of free group automorphisms, necessary and sufficient conditions for hyperbolicity

Outline of Part II

- Prerequisites: (il)legal turns, legal paths, Perron-Frobenius-metrics, bounded cancellation lemma

- Relative train tracks: definition, existence

- Application: invariant actions on $\mathbb{R}$-trees, Scott conjecture

References

- Bestvina-Handel: Train tracks and automorphisms of free groups, *Annals of Mathematics*, 1992

- Gaboriau-Levitt-Lustig: A dendrological proof of the Scott conjecture for automorphisms of free groups, *Proc Edinburgh Math. Soc.*, 1998

- Lustig: Automorphisms of free groups I: Perron-Frobenius actions on $\mathbb{R}$-trees

Train track maps

Recall:

Definition. A homotopy equivalence $f : G \to G$ inducing an outer automorphisms $\mathcal{O}$ of $\pi_1 G$ is a *train track map* if for every $n \geq 1$, f^n is a topological representative of $\mathcal{O}$. In particular, for every positive exponent n, the restriction of f^n to the interior of each edge is an immersion.

Theorem. *Irreducible automorphisms have train track representatives.*

Legal paths, illegal turns

Let $f : G \to G$ be a train track map. A *turn* is a pair (E_1, E_2) of two edges emanating from the same vertex. A turn is *illegal* if for some exponent $k > 0$, $f^k(\bar{E}_1 E_2)$ is not an immersion (*legal* otherwise).

An edge path in G is a *legal path* if it contains no illegal turns.

Note that the train track property implies that f maps every edge to a legal path. However, there may still be illegal turns!

Perron-Frobenius metrics

Let $f : G \to G$ be a train track map with irreducible transition matrix M. Then M has a PF-eigenvalue λ and a corresponding positive eigenvector $v = (v_1, ..., v_n)$, where n is the number of edges of G.

Define a PF-metric on G by letting $L(E_i) = v_i$.

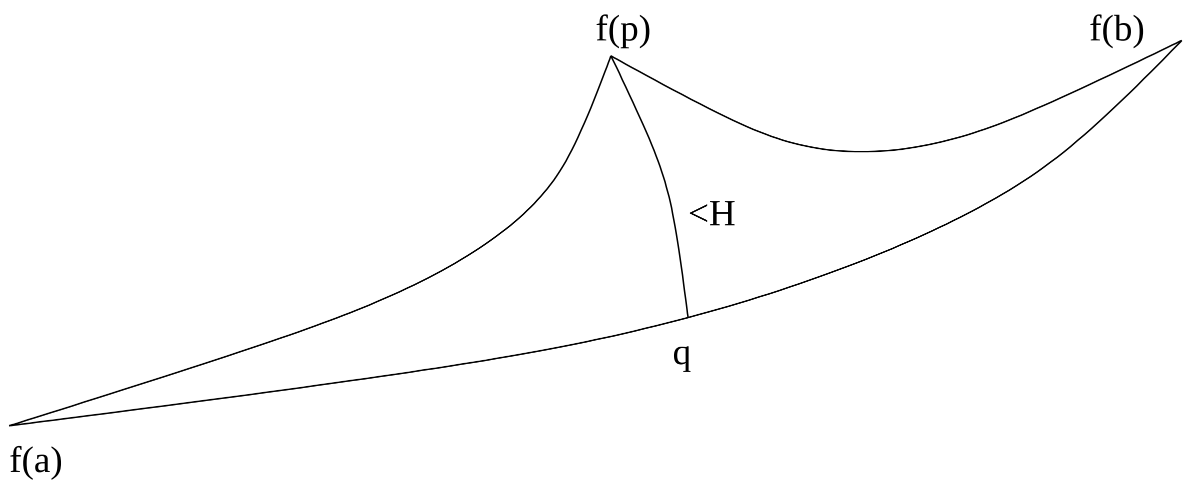
Properties of PF-metrics:

- If ρ is a legal path, then $L(f(\rho)) = \lambda L(\rho)$.
- In general, $L(f(\rho)) \leq \lambda L(\rho)$.

Bounded cancellation lemma

Lemma (Bounded cancellation lemma for homot
Let $f : G \to G$ be a homotopy equivalence of a finite graph G. There exists a constant C_f, depending only on f, with the property that for any path ρ in G obtained by concatenating two paths α, β, we have

$$L([f(\rho)]) \geq L([f(\alpha)]) + L([f(\beta)]) - C_f.$$



Exponential growth versus bounded cancellation

Important observation: If $\lambda > 1$ and ρ is path with a sufficiently long legal subpath ρ', then the length of the segment of $f^n(\rho)$ corresponding to ρ' will always grow exponentially (as a function of n).

This motivates the notion of a *critical length*

$$L_C = \frac{2C_f}{\lambda - 1}$$

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Filtrations and strata

Let $f : G \to G$ be a topological representative with reducible transition matrix M. Then, after relabelling the edges of G if necessary, M is a block upper triangular matrix

$$M = \begin{pmatrix} M_1 & & * \\ & \cdots & \\ 0 & & M_s \end{pmatrix},$$

where the blocks M_i on the diagonal are irreducible or 0.

This decomposition of M gives rise to a filtration

$$\{\} = G_0 \subset G_1 \subset \ldots \subset G_s = G$$

of G. The subgraphs $H_i = G_i - G_{i-1}$ are called *strata*. Note that each stratum M_i has a corresponding matrix M_i.

Classification of strata

There are three kinds of strata:

- A stratum H_i is *of exponential growth* if M_i is irreducible with $\lambda > 1$.

- A stratum H_i is *of polynomial growth* if M_i is irreducible with $\lambda = 1$.

- A stratum H_i is a *zero stratum* if $M_i = (0)$.

Relative train track maps

A topological representative $f : G \to G$ is a *relative train track map* if the following conditions are satisfied for all exponentially growing strata H_r:

1. If E is an edge in H_r, then the first and last edges of the edge path $f(E)$ lie in H_r. In particular, if (E_1, E_2) is a turn with E_1 in G_r and E_2 in G_{r-1}, then (E_1, E_2) is legal (read: mixed turns are legal).

2. If ρ is a nontrivial path in G_{r-1} with endpoints in $H_r \cap G_{r-1}$, then $f(\rho)$ is homotopic (rel endpoints) to a nontrivial path with endpoints in $H_r \cap G_{r-1}$.

3. If E is an edge in H_r, then $f(E)$ contains no illegal turns in H_r (read: $f(E)$ is r-legal).

Existence of relative train track maps

Theorem. *Every outer automorphism of a finitely generated free group F_n can be represented by a relative train track map.*

Remark. A train track map representing an irreducible automorphism is a relative train track map with one stratum.