Train Tracks and Applications, Part I

Peter Brinkmann

September 7, 2000

Outline

1. Part I: Train tracks for irreducible automorphisms: definitions, train track algorithm, application to surface homeomorphisms

2. Part II: Train tracks for reducible automorphisms: relative train track maps, improved relative train track maps, actions on R-trees, Scott conjecture

3. Part III: Hyperbolic automorphisms: mapping tori of free group automorphisms, necessary and sufficient conditions for hyperbolicity

Outline of Part I

- Introduction: automorphisms of free groups, homotopy equivalences of graphs, definition of train tracks

- Computing train tracks: irreducible matrices, Perron-Frobenius theorem, irreducible automorphisms, train track algorithm

- Application: pseudo-Anosov homeomorphisms of surfaces

References

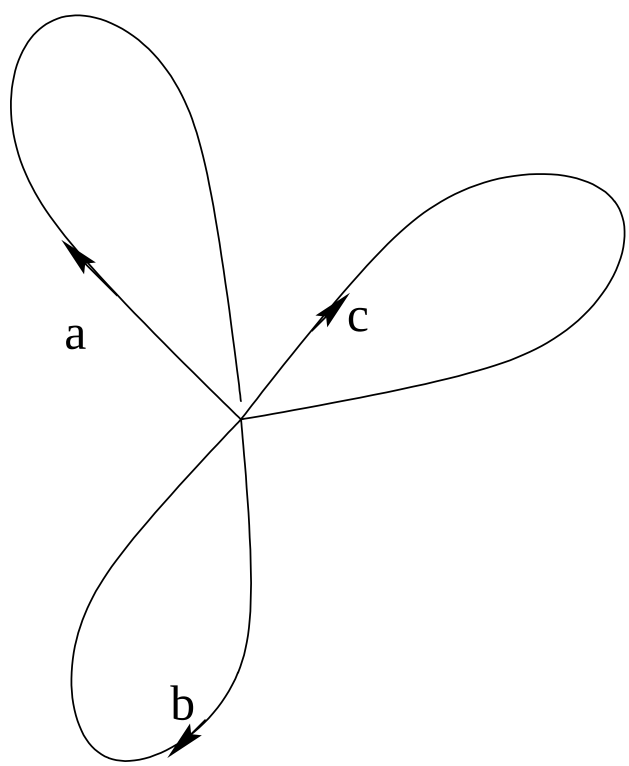
- Bestvina-Handel: Train tracks and automorphisms of free groups, *Annals of Mathematics*
- Bestvina-Handel: Train tracks for surface homeomorphisms, *Topology*
- B.: An implementation of the Bestvina-Handel algorithm for surface homeomorphisms, *Experimental Mathematics*

Homotopy equivalences of graphs

- The fundamental group of a finite graph G is a finitely generated free group F.

- An automorphism ϕ of F can be represented as a homotopy equivalence of G (fixing a base point).

- An *outer* automorphisms $\mathcal{O}$ of F can be represented as a homotopy equivalence of G.



Train track maps (definition)

Let G be a finite graph without vertices of valence one or two.

A *topological representative of* $\mathcal{O}$ is a homotopy equivalence $f : G \to G$ if

1. it maps vertices to vertices and
2. for every edge E of G, the restriction of f to the interior of E is an immersion and
3. there are no invariant forests

Definition. A homotopy equivalence $f : G \to G$ is a *train track map* if for every $n \geq 1$, f^n is a topological representative.

Positive automorphisms immediately give rise to train track maps.

Perron-Frobenius theory

Let M be an $n \times n$ matrix with nonnegative integer entries. M is *irreducible* if for every tuple i, j, there exists some exponent $m \geq 1$ such that the ij-th entry of M^m is positive.

Theorem. *Let M be an irreducible matrix. Then M has a positive, real eigenvalue λ with the following properties:*

1. *$\lambda \geq 1$ with equality iff M is a transitive permutation matrix.*

2. *If μ is an eigenvalue of M, then $\lambda \geq |\mu|$.*

3. *The eigenspace V has dimension one. Moreover, V is spanned by a positive vector.*

4. *Lots of other nice properties...*

The eigenvalue λ is called the *PF-eigenvalue* of M.

Irreducible automorphisms

Let $f : G \to G$ be a topological representative of some outer automorphisms $\mathcal{O}$ of F, and let $E_1, \ldots, E_k$ denote the edges of G. The *transition matrix* M of f is the $k \times k$ matrix $M = (a_{i,j})$, where $a_{i,j}$ is the number of times the image of E_j crosses E_i (regardless of orientation).

Definition. An outer automorphisms $\mathcal{O}$ is said to be *irreducible* if the transition matrix of every topological representative of $\mathcal{O}$ is irreducible.

Important example: A pseudo-Anosov automorphism of a surface S with one puncture induces an irreducible automorphism of $\pi_1 S$.

Existence of train tracks

Theorem. *Irreducible automorphisms have train track representatives.*

Strategy of proof: Find an algorithm that takes any homotopy equivalence $f : G \to G$ representing some automorphism $\mathcal{O}$ and improves it step by step until either

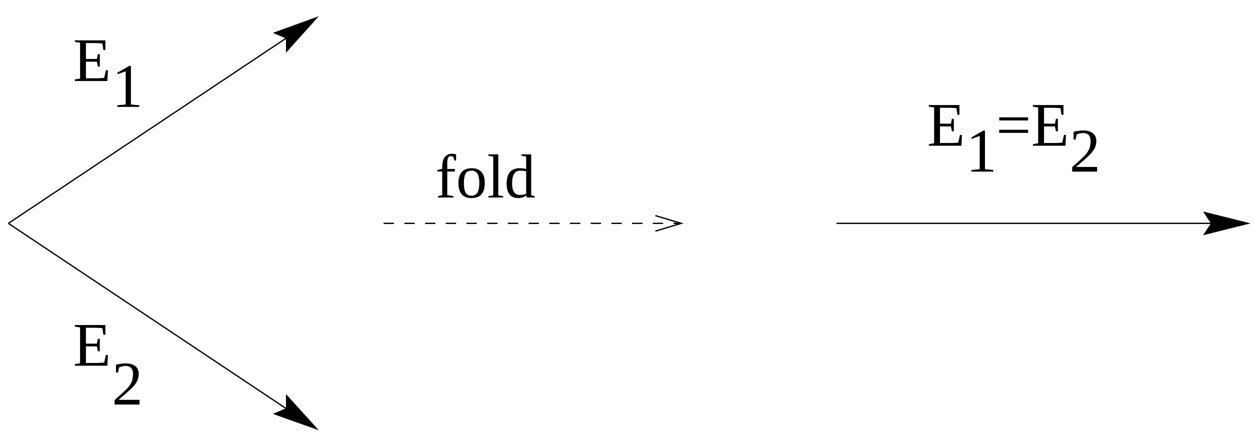
- it finds a train track map *or*
- it finds a representative of $\mathcal{O}$ with reducible transition matrix.

One of the two possibilities will occur after finitely many steps, and we will use the PF-eigenvalue of the transition matrix as a measure of progress.

Folding graphs*

Given a finite graph G with two edges E_1, E_2 emanating from the same vertex, we construct a new graph G' by identifying E_1 and E_2. We say that G' is obtained from G by *folding*.

There exists a natural projection $p : G \twoheadrightarrow G'$.



Moreover, if we have a map $f : G \to G$ with $f(E_1) = f(E_2)$, then we obtain an induced map $f' : G' \to G'$.

*John R. Stallings, Topology of Finite Graphs, *Invent. math. 71, 551-565 (1983)*

Moves of the algorithm

1. Subdivisions: prepare for fold

2. *Fold: main move*

3. Tightening: get rid of backtracking

4. Valence-1-homotopy, valence-2-homotopy: remove vertices of valence one or two

5. Collapsing invariant forests

Main idea: If $f : G \to G$ fails to be a train track map, then (after preliminary subdivision) we can fold G. After tightening, ..., the resulting map $f' : G' \to G'$ is either reducible or irreducible with strictly smaller PF-eigenvalue.

Surface homeomorphisms

Let S be a surface with exactly one puncture. If $g : S \to S$ is a pseudo-Anosov homeomorphism of S, then g induces an irreducible outer automorphism $\mathcal{O}$ of $\pi_1 S$.

Hence, we can find a train track map $f : G \to G$ representing g_*, and the train track map allows us to read off the following data:

- the pA growth rate of g
- the singularities of the stable/unstable foliation respected by g

Moreover, given any homeomorphism h of S, we can determine whether h is pA.