Math 510 - Fall 2006
Algebraic Curves and Riemann Surfaces

Problem Set 3

Question 1
Let $f$ be the meromorphic function on $\mathbb{P}^1$ defined by the rational function

$$z \mapsto \frac{z^3}{(1-z^2)},$$

and let $F : \mathbb{P}^1 \to \mathbb{P}^1$ be the corresponding holomorphic map. In Problem Set 2 you computed the degree of $F$ - and hopefully found that it is 3!. Find all the zeros and poles of $f$, and all the branch points and ramification points of $F$. Verify by explicit computation that $\sum \nu_p(f) = 0$, where the sum is over all points in $\mathbb{P}^1$, and that the Riemann-Hurwitz formula holds for $F$.

Question 2
Prove that any intersection of algebraic subsets of $\mathbb{P}^n$ is algebraic. Prove that a finite union of algebraic subsets of $\mathbb{P}^n$ is algebraic.

Question 3
Let $L \subset \mathbb{P}^n$ be a projective linear subspace of dimension $k$. Prove that there is a linear automorphism $T : \mathbb{P}^n \to \mathbb{P}^n$ such that

$$T(L) = \{ [z_0, z_1, \ldots, z_k, \ldots, z_n] \mid z_{k+1} = \cdots = z_n = 0 \} = \mathbb{P}^k.$$
Question 4

Prove that any two irreducible conics (i.e. degree 2 curves in \( \mathbb{P}^2 \)) are projectively equivalent. Follow these steps:

1. Assume that \([0, 1, 0]\) is a smooth point. Show that after a linear transformation the defining equation can be put in the form

\[
F(x, y, z) = ax^2 + bz^2 + cxy + dyz
\]

2. Find a projective transformation which takes the curve to the curve

\[
x^2 = yz .
\]

Question 5

Show that the subset of \( \mathbb{C}^2 \) consisting of all points

\[
(t^2, t^3 + 1), \ t \in \mathbb{C}
\]
is an algebraic curve.

Question 6

Find the singular points of the following curves in \( \mathbb{C}^2 \):

(a) \( y^2 = x^3 - x \)
(b) \( y^3 - y^2 + x^3 - x^2 + 3y^2x + 3x^2y + 2xy = 0 \)

Find the tangent lines at the smooth points.

Question 7

The multiplicity of a point \([a, b, c]\) on a projective curve in \( \mathbb{P}^2 \) defined by \( F(x, y, z) = 0 \) is the smallest integer \( m \) such that

\[
\frac{\partial^m F}{\partial x^i \partial y^j \partial z^k}(a, b, c) \neq 0
\]

for some \( i, j, k \) such that \( i + j + k = m \). Find the singular points and their multiplicities for the following projective curves:

(a) \( xy^4 + yz^4 + xz^4 = 0 \)
(b) \( y^2z = x(x - z)(x - \lambda z), \ \lambda \in \mathbb{C} \)
Question 8

Exercise 10.3 on page 49 of Griffiths.